Right-invariant metrics on diffeomorphisms groups with applications to diffeomorphic image registration.

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Outline

Motivation

- Developing geometrical and statistical tools to analyse biomedical shapes distributions/evolutions,
- Developing the associated numerical algorithms.

Example of problems of interest

Given two shapes, find a diffeomorphism of $\mathbb{R}^3$ that maps one shape onto the other

Different data types and different way of representing them.

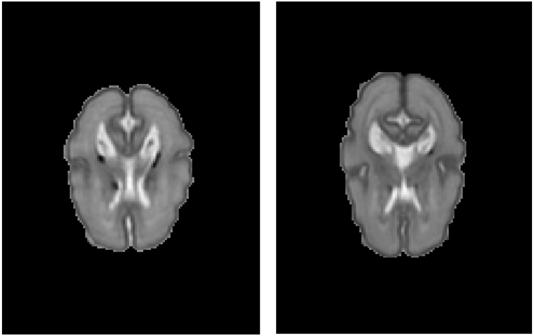


Figure: Two slices of 3D brain images of the same subject at different ages

About Computational Anatomy

Old problems:

1. to find a framework for registration of biological shapes,
2. to develop statistical analysis in this framework.

Action of a transformation group on shapes or images
Idea pioneered by Grenander and al. (80's), then developed by M.Miller, A.Trouvé, L.Younes.

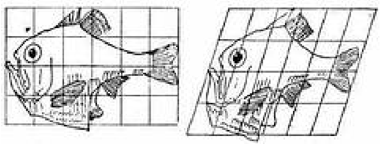


Figure: deforming the shape of a fish by D'Arcy Thompson, author of *On Growth and Forms* (1917)

New problems like study of Spatiotemporal evolution of shapes within a diffeomorphic approach

A Riemannian approach to diffeomorphic registration

Several diffeomorphic registration methods are available:

- Free-form deformations B-spline-based diffeomorphisms by D. Rueckert
- Log-demons (X.Pennec et al.)
- Large Deformations by Diffeomorphisms (M. Miller,A. Trouvé, L. Younes)

Only the last one provides a Riemannian framework.

- $v_t \in V$ a time dependent vector field on $\mathbb{R}^n$.
- $\phi_t \in Diff$, the flow defined by

$$\partial_t \phi_t = v_t(\phi_t)\,. \tag{1}$$

Action of the group of diffeomorphism G_0 (flow at time 1):

$$\Pi : G_0 \times \mathcal{C} \to \mathcal{C}\,,$$
$$\Pi(\phi, X) \doteq \phi.X$$

Right-invariant metric on G_0: $d(\phi_{0,1}, \mathrm{Id})^2 = \frac{1}{2}\int_0^1 |v_t|_V^2 dt$.
$\longrightarrow$ Strong metric needed on V
(Mumford and Michor: *Vanishing Geodesic Distance on...*)

Left action and right-invariant metric

Definition (Left action)

A left action for the group G is a map $G \times M \to M$ satisfying

1. $Id \cdot q = q$ for $q \in M$.
2. $g_2 \cdot (g_1 \cdot M) = (g_2 g_1) \cdot q$.

Example: The group on itself, $GL_n(\mathbb{R})$ acting by multiplication on $\mathbb{R}^n$.

Definition (Right-invariant length)

Let G be a Lie group and $\| \cdot \|$ a scalar product on its Lie algebra $\mathfrak{g} := T_{Id} G$. Let $g(t)$ be a C^1 path on the group. The length of the path $\ell(g(t))$ can be defined by:

$$\ell(g(t)) = \int_0^1 \|\partial_t g(t) g(t)^{-1}\|^2 \, dt \,. \tag{2}$$

Note that $v(t) = \partial_t g(t) g(t)^{-1} \in \mathfrak{g}$. This is called right-trivialized velocity.

Left action and right-invariant metric

Definition (Right-invariant metric)

Let $g_1, g_2 \in G$ be two group elements, the distance between g_1 and g_2 can be defined by:

$$d^2(g_1, g_2) = \inf_{g(t)} \left\{ \int_0^1 \|\partial_t g(t) g(t)^{-1}\|^2 \, dt \,|\, g(0) = g_1 \text{ and } g(1) = g_1 \right\}$$

Minimizers are called geodesics.

Right-invariance simply means:

$$d^2(g_1 g, g_2 g) = d(g_1, g_2) \,.$$

It comes from:

$$\partial_t(g(t)g_0)(g(t)g_0)^{-1} = \partial_t g(t) g_0 g_0^{-1} g(t)^{-1} = \partial_t g(t) g(t)^{-1} \,.$$

Euler-Poincaré equation

Compute the Euler-Lagrange equation of the distance functional:

$$\frac{\partial L}{\partial g} - \frac{d}{dt}\frac{\partial L}{\partial \dot{g}} = 0$$

With a change of variable, let's do "reduction" on the Lie algebra: Special case of $\int_0^1 L(g, \dot{g})dt = \int_0^1 \ell(v(t), Id)dt$.

$$(\partial_t + \mathrm{ad}_v^*)\frac{\partial \ell}{\partial v} = 0\,.$$

Proof.

Compute variations of $v(t)$ in terms of $u(t) = \delta g(t) g(t)^{-1}$. Find that admissible variations on $\mathfrak{g}$ can be written as:
$\delta v(t) = \dot{u} - \mathrm{ad}_v\, u$ for any u vanishing at 0 and 1. Recall that $\mathrm{ad}_v\, u = [u, v]$. □

EPDiff equation

Let's formally apply this to the group of diffeomorphisms of $\mathbb{R}^d$ with a metric $\langle u, v\rangle = \langle u, Lv\rangle_{L^2}$. Denoting $m = Lu$,

$$\partial_t m + Dm.u + Du^T.m + \mathrm{div}(u)m = 0\,. \tag{3}$$

For example, the L^2 metric gives:

$$\partial_t m + Du.u + Du^T.u + \mathrm{div}(u)m = 0\,. \tag{4}$$

On the group of volume preserving diffeomorphisms of (M, μ) with the L^2 metric:
Euler's equation for ideal fluid where $\mathrm{div}(u) = 0$

$$\partial_t u + \nabla_u u = -\nabla p\,,$$

(use $\mathrm{div}(u) = 0$ and write the term $Du^T.u$ as a gradient as $\nabla \|u\|_{L^2}$)
Other equations: Camassa-Holm equation, Hunter-Saxton equation...

Left action and momentum map

Suppose that the action is transitive and a submersion at identity, if $g(0) = Id$

$$v \in \mathfrak{g} \to v.q := \frac{d}{dt}_{|0} g(t) \cdot q$$

surjective. Define a Riemannian metric on TM by:

$$\|\delta q\|^2 = \inf_{v \in \mathfrak{g}} \{ \|v\|^2 \,|\, v \cdot q = \delta q \} \,.$$

Definition

Let $p \in T_q^* M$ be a co-tangent vector at q then the momentum map is

$$J : T^* M \to \mathfrak{g}^*$$
$$(q, p) \to \langle J(q, p), v \rangle_{\mathfrak{g}} = (p, v \cdot q)$$

A Riemannian framework on the orbit

Proposition

Right-invariant metric + left action $\implies$ Riemannian metrics on the orbits. The map $\Pi_{q_0} : G \ni g \mapsto g \cdot q_0 \in Q$ is a Riemannian submersion.

Proposition

The inexact matching functional

$$\mathcal{J}(v) = \int_0^1 |v_t|_V^2 dt + \frac{1}{\sigma^2} d(\phi_1.q_0, q_{target})^2$$

leads to geodesics on the orbit of A for the induced Riemannian metric.

Optimal control viewpoint

Move q_0 in order to minimize $\frac{1}{2}\int_0^1 \|v\|^2 dt + \frac{1}{2}\|q(1) - q_{target}\|^2$ under the constraint $\dot{q} = v \cdot q$.

Pontryagin principle implies extremals are solutions of:

$$\dot{q} = v \cdot q \tag{5}$$

$$\dot{p} = -v^* \cdot p \tag{6}$$

$$Lv = J(q,p)\,. \tag{7}$$

and $p(1) + \partial_q[\frac{1}{2}\|q - q_{target}\|^2](q(1)) = 0$.

Proposition

$J(q,p)$ satisfies the EPDiff equation.

Matching problems in a diffeomorphic framework

1. U a domain in $\mathbb{R}^n$
2. V a Hilbert space of C^1 vector fields such that:

$$\|v\|_{1,\infty} \leq C|v|_V .$$

V is a Reproducing kernel Hilbert Space (RKHS): (pointwise evaluation continuous)
$\Longrightarrow$ Existence of a matrix function k_V (kernel) defined on $U \times U$ such that:

$$\langle v(x), a \rangle = \langle k_V(., x)a, v \rangle_V .$$

Right invariant distance on G_0

$$d(\mathrm{Id}, \phi)^2 = \inf_{v \in L^2([0,1],V)} \int_0^1 |v_t|_V^2 dt \,,$$

$\longrightarrow$ geodesics on G_0.

Matching problems in a diffeomorphic framework

Action of G_0 on group of points (Landmarks):

$$\phi.(x_1, \dots, x_k) = (\phi(x_1), \dots, \phi(x_k)) \, ,$$

Momentum map: $\sum_{i=1}^{k} \delta_{x_i}^{p_i}$.

Action of G_0 on images ($I \in L^\infty(U)$):

$$\phi.I = I \circ \phi^{-1} \, .$$

Momentum map: $J(I, P) = -P\nabla I$.

Action of G_0 on surfaces:

$$\phi.S = \phi(S) \, ,$$

Action on measures:

$$(\phi.\mu, f) \doteq (\mu, f \circ \phi)$$

Generalized to currents (linear form on $\Omega_c^k(\mathbb{R}^d)$) and varifolds.

Inexact matching: taking noise into account

Minimizing

$$\mathcal{J}(v) = \frac{1}{2}\int_0^1 |v_t|_V^2 dt + \frac{1}{2\sigma^2} d(\phi_{0,1}.A, B)^2 \,.$$

In the case of landmarks:

$$\mathcal{J}(\phi) = \frac{1}{2}\int_0^1 |v_t|_V^2 dt + \frac{1}{2\sigma^2} \sum_{i=1}^{k} \|\phi(x_i) - y_i\|^2 \,,$$

In the case of images:

$$d(\phi_{0,1}.I_0, I_{target})^2 = \int_U |I_0 \circ \phi_{1,0} - I_{target}|^2 dx \,.$$

Existence of minimizers: weak convergence in $L^2([0,1], V) \implies$ uniform convergence of the flow (on compact sets).

Numerical solutions

How to numerically compute a solution? Steepest gradient descent on:

1. $L^2([0,1],V)$, i.e. the functional $\mathcal{J}$ itself,
2. the subspace of Euler-Lagrange solutions.

Gradient of

$$\mathcal{J}(v) = \frac{1}{2}\int_0^1 |v_t|_V^2 dt + \frac{1}{2\sigma^2}\int_U |I_1 - I_{target}|^2 dx \,.$$

under the constraint $\partial_t I(t) + \langle v, \nabla I \rangle = 0$.
Introduce Lagrange multipliers:

$$\mathcal{J}(v) = \frac{1}{2}\int_0^1 |v_t|_V^2 dt + \frac{1}{2\sigma^2}\int_U |I_1 - I_{target}|^2 dx + \int_0^1 P(t)\left(\partial_t I_t + \langle v, \nabla I \rangle\right) dt \,.$$

The gradient is given by:

$$\begin{aligned}
&\nabla \mathcal{J}(v)(t) = v(t) + K \star (P(t)\nabla I(t)) \\
&\partial_t I(t) + \langle v, \nabla I \rangle = 0 \\
&\partial_t P + \nabla \cdot (Pv) = 0 \\
&P(1) + \frac{1}{2\sigma^2}(I_1 - I_{target}) = 0 \,.
\end{aligned}$$

Shooting methods

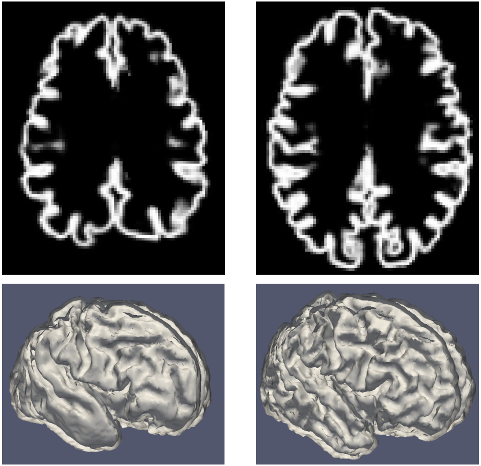
Reduction of the functional J to Euler-Lagrange solutions:
The matching functional can be rewritten on the geodesic flow as:

$$\mathcal{S}(P(0)) = \frac{\lambda}{2}\langle \nabla I(0)P(0), K \star \nabla I(0)P(0)\rangle_{L^2} + \frac{1}{2}\|I(1) - J\|^2_{L^2} \,. \tag{8}$$

with:

$$\begin{cases} \partial_t I + v \cdot \nabla I = 0, \\ \partial_t P + \nabla \cdot (vP) = 0, \\ v + K \star (P \nabla I) = 0. \end{cases} \tag{9}$$

Experiments



Source image G_{33} **Target image** G_{36}

Figure: Registered segmented cortex out of MR images of pre-born babies at 33 and 36 weeks of gestational age, denoted by G_{33} and G_{36} respectively. **(Top)** Slices out of the segmented images. **(Bottom)** Internal face of the volumetric images surface.

Practical issues

Main issues for practical applications:

- choice of the metric (prior): mixture of Gaussian kernels:

$$K(x, y) = \sum_{i=1}^{n} \beta_i e^{-\frac{\|x-y\|^2}{\sigma_i^2}} \tag{10}$$

- choice of the similarity measure.

Ad-hoc solutions for the first problem: Mixture of Gaussian kernels:

Link with optimal transport

L^2 distance on the group of diffeomorphisms + left action on densities $\implies$ Riemannian submersion. Minimization of:

$$\int_0^1 \|\partial_t \phi\|^2_{L^2(\rho_0)}\, dt = \int_0^1 \int_M \|v(x)\|^2 \rho(x)\, dx\, dt$$

under the constraint: $\rho_1 = \phi_*(1)(\rho_0)$.
Geodesic equations:

$$\begin{cases} \dot{\rho} + \nabla \cdot (\rho \nabla P) = 0 \\ \dot{P} + \frac{1}{2}|\nabla P|^2 = 0 \end{cases} \tag{11}$$

For LDDMM:

- Due to smoothness, disjoint orbits of measures.
- No convexity.
- No scale invariance.

Why does the Riemannian framework matter?

Generalizations of statistical tools in Euclidean space:

- Distance often given by a Riemannian metric.
- Straight lines $\to$ geodesic defined by

$$\text{Variational definition: } \arg\min_{c(t)} \int_0^1 \|\dot{c}\|^2_{c(t)}\, dt = 0\,,$$

$$\text{Equivalent (local) definition: } \nabla_{\dot{c}}\dot{c} = \ddot{c} + \Gamma_{i,j}(\dot{c},\dot{c}) = 0\,.$$

- Average $\to$ Fréchet/Kärcher mean.

$$\text{Variational definition:} \qquad \arg\min\{x \to E[d^2(x,y)]d\mu(y)\}$$

$$\text{Critical point definition:} \qquad E[\nabla_x d^2(x,y)]d\mu(y)] = 0\,.$$

- PCA $\to$ Tangent PCA or PGA.
- Geodesic regression, cubic regression...(variational or algebraic)

Riemannian metric needed, or at least a connection.

Pitfalls:

- Loose uniqueness of geodesic or average (positive curvature).
- Equivalent definitions diverge (generalisation of PCA).

Kärcher mean on 3D images

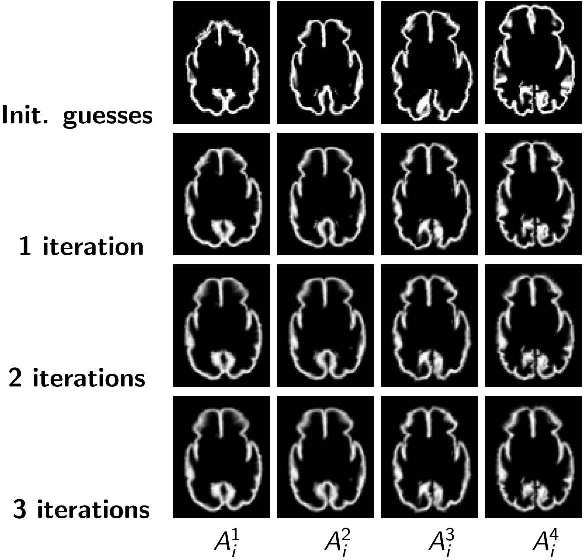


Figure: Average image estimates A_i^m, $m \in \{1, \cdots, 4\}$ after i =0, 1, 2 and 3 iterations.

Bayesian interpretation

The prior in the functional

$$\mathcal{J}(v) = \int_0^1 |v_t|_V^2 dt + \frac{1}{\sigma^2} d(\phi_{0,1}.A, B)^2$$

suggests a white noise in time for generic evolutions.

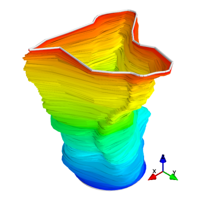


Figure: Kunita flows

$\rightarrow$ Not realistic for evolutions of biological shapes.

Interpolating sparse longitudinal shape data

What we aim to do:

Within a diffeomorphic framework:

Let $(S^i_{t^i_0}, \dots, S^i_{t^i_k})_{i \in [1,n]}$ be a $n-$sample of shape sequences indexed by the time $(t^i_0, \dots, t^i_n) \subset [0, 1]$.

Having in mind biological shapes, at least two problems

◇ To find a deterministic framework to treat each sample.
(in which space to study these data?)

◇ To develop a probabilistic framework to do statistics.
(classification into normal and abnormal growth)

A natural attempt

How to interpolate a sequence of data $(S_0, \ldots, S_{t_k})$ (images, surfaces, landmarks ...)

When $k = 1 \longrightarrow$ standard registration problem of two images: Geodesic on a diffeomorphism group - LDDMM framework (M.Miller, A.Trouvé, L.Younes, F.Beg,...)

$$\mathcal{F}(v) = \frac{1}{2}\int_0^1 |v_t|_V^2\, dt + |\phi_1.S_0 - S_{t_1}|^2\,,$$

$$\begin{cases} \phi_0 = Id \\ \dot{\phi}_t = v_t(\phi_t)\,. \end{cases} \tag{12}$$

Extending it to $k > 1$,

$$\mathcal{F}(v) = \frac{1}{2}\int_0^{t_k} |v_t|_V^2\, dt + \sum_{j=1}^{k} |\phi_{t_j}.S_0 - S_{t_j}|^2\,,$$

$\Longrightarrow$ piecewise geodesics in the group of diffeomorphisms

Illustration on 3D images

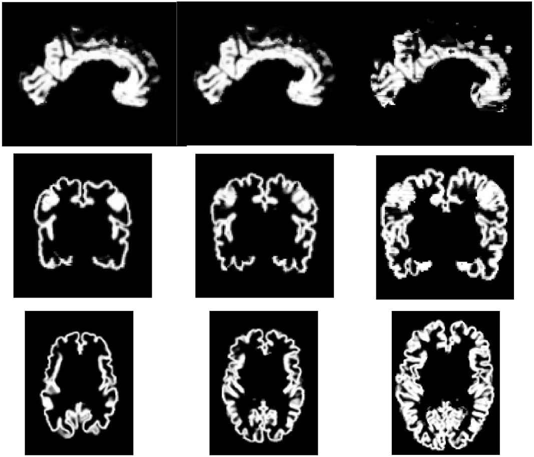


Figure: Slices of 3D volumic images: 33 / 36 / 43 weeks of gestational age of the same subject.

How to smoothly interpolate longitudinal data

In the Euclidean space:

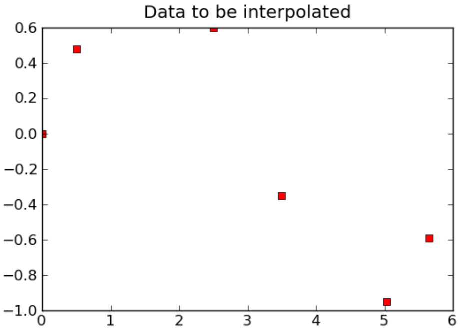


Figure: Sparse data from a sinus curve

Minimizing the L^2 norm of the **speed** $\rightarrow$ piecewise linear interpolation

Linear interpolation

What is acceleration in our context?

First attempt, on the group in the matching functional:

$$\mathcal{F}(v) = \frac{1}{2}\int_0^1 |v_t|_V^2 \, dt + |\phi_1.S_0 - S_{t_1}|^2 \,, \tag{13}$$

Replace the L^2 norm of the speed:

$$\frac{1}{2}\int_0^1 |v_t|_V^2 \, dt \tag{14}$$

by the L^2 norm of the acceleration of the vector field:

$$\frac{1}{2}\int_0^1 |\frac{d}{dt} v_t|_V^2 \, dt + |\phi_1.S_0 - S_{t_1}|^2 \,, \tag{15}$$

Null cost for this norm $\longrightarrow v_t \equiv v_0$: Incoherent

Correct notion of acceleration

Acceleration on a Riemannian manifold M: let $c : I \to M$ be a C^2 curve. The notion of acceleration is:

$$\frac{D}{dt}\dot{c}(t) = \nabla_{\dot{c}}\dot{c}(= \ddot{c}_k + \sum_{i,j} \dot{c}_i \Gamma^k_{i,j} \dot{c}_j) \tag{16}$$

with ∇ the Levi-Civita connection.

Riemannian splines: Crouch, Silva-Leite (90's)

$$\text{On } SO(3) \ \inf_c \int_0^1 \frac{1}{2} |\nabla_{\dot{c}_t} \dot{c}_t|^2_M dt\,. \tag{17}$$

subject to $c(i) = c_i$ and $\dot{c}(i) = v_i$ for $i = 0, 1$.

Elastic Riemannian splines:

$$\inf_c \int_0^1 \frac{1}{2} |\nabla_{\dot{c}_t} \dot{c}_t|^2_M + \frac{\alpha}{2} |\dot{c}_t|^2_M dt\,. \tag{18}$$

subject to $c(i) = c_i$ and $\dot{c}(i) = v_i$ for $i = 0, 1$.

A modeling question

The Euler-Lagrange equation for Riemannian cubics is

$$\nabla^3_{\dot c}\dot c + R(\nabla_{\dot c}\dot c, \dot c)\dot c = 0\,, \tag{19}$$

where R is the curvature tensor of the metric.

Remarks

If $\pi : M \mapsto B$ is a Riemannian submersion then:
geodesics lift to geodesics.
Probably not true for Riemannian cubics ...

In our context of a group action, $G \times M \mapsto M$:
$\Pi_{q_0} : G \ni g \mapsto g \cdot q_0 \in Q$ is a Riemannian submersion

Question

Higher-order on the group (upstairs) or higher-order on the orbit (downstairs)?

The convenient Hamiltonian setting

Hamiltonian equations of geodesics for landmarks:

$$\textbf{Geodesics} \quad \begin{cases} \dot{p} = -\partial_q H(p,q) = -[J(q,p)^\sharp]^* \cdot p \\ \dot{q} = \partial_p H(p,q) = J(q,p)^\sharp \cdot q \end{cases} \tag{20}$$

with $H(p,q) = H(p_1,\dots,p_n,q_1,\dots,q_n) \doteq \frac{1}{2}\sum_{i,j=1}^n p_i k(q_i,q_j) p_j$ and k is the kernel for spatial correlation.

Lemma

On a general Riemannian manifold,

$$\nabla_{\dot{q}}\dot{q} = K(q)(\dot{p} + \partial_q H(p,q)) \tag{21}$$

where $\dot{q} = K(q)p$ with $K(q)$ being the identification given by the metric between $T_q^* Q$ and $T_q Q$.

Splines on shape spaces

We introduce a forcing term u as:

Perturbed geodesics
$$\begin{cases} \dot{p}_t = -\partial_q H(p_t, q_t) + u_t \\ \dot{q}_t = \partial_p H(p_t, q_t) \end{cases} \tag{22}$$

Definition (Shape Splines)

Shape splines are defined as minimizer of the following functional:

$$\inf_u J(u) \doteq \frac{1}{2} \int_0^{t_k} \|u_t\|_X^2 \, dt + \sum_{j=1}^{k} |q_{t_j} - x_{t_j}|^2 \,. \tag{23}$$

subject to (q, p) perturbed geodesic through u_t for a freely chosen norm $\| \cdot \|_X$ on T_q^*.

Simulations

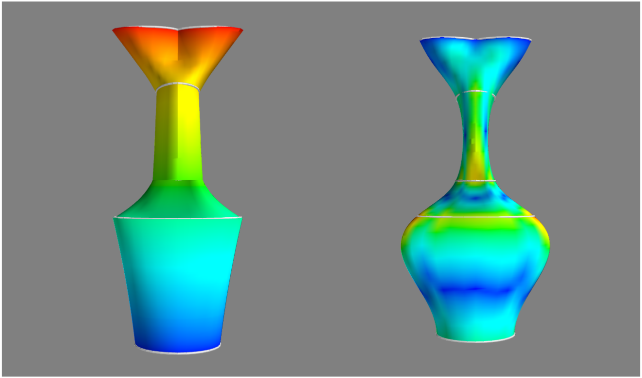


Figure: Comparison between piecewise geodesic interpolation and spline interpolation

- Matching of 4 timepoints from an initial template.
- $|\cdot|_X$ is the Euclidean metric.
- Smooth interpolation in time.

Information contained in the acceleration and extrapolation

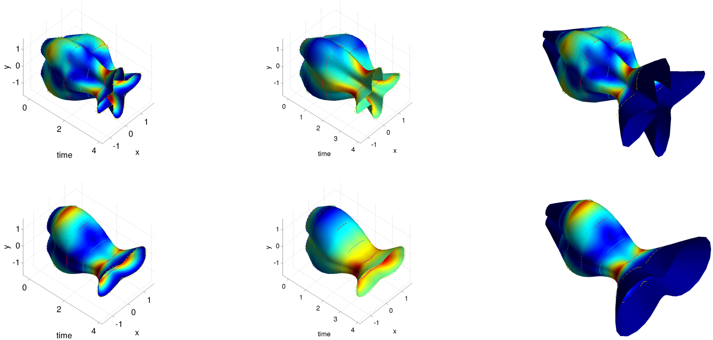


Figure: On each row: two different examples of the spline interpolation. In the first column, the norm of the control is represented whereas the signed normal component of the control is represented in the second one. The last column represents the extrapolation.

Robustness to noise

Due to the spatial regularisation of the kernel:

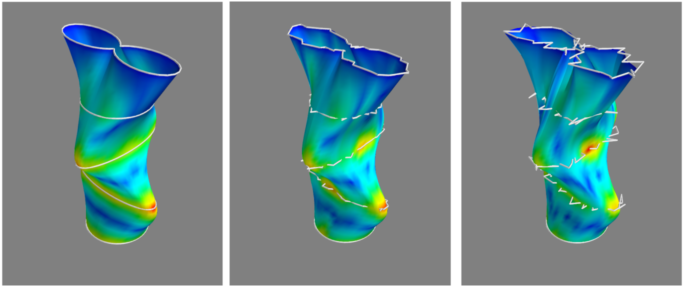


Figure: Gaussian noise added to the position of 50 landmarks

- Left: no noise.
- Center: standard deviation of 0.02.
- Right: standard deviation of 0.09.

A generative model of shape evolutions

A stochastic model:

Theorem

If k is C^1, the solutions of the stochastic differential equation defined by

$$\begin{cases} dp_t = -\partial_x H_0(p_t, x_t)dt + u_t(x_t)dt + \varepsilon(p_t, x_t)dB_t \\ dx_t = \partial_p H_0(p_t, x_t)dt. \end{cases} \tag{24}$$

are non exploding with few assumptions on u_t and ε.

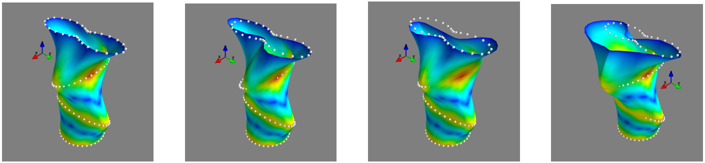


Figure: The first figure represents a calibrated spline interpolation and the three others are white noise perturbations ot the spline interpolation with respectively $\sqrt{n}\epsilon$ set to $0.25, 0.5$ and 0.75.

Simple PCA on the forcing term

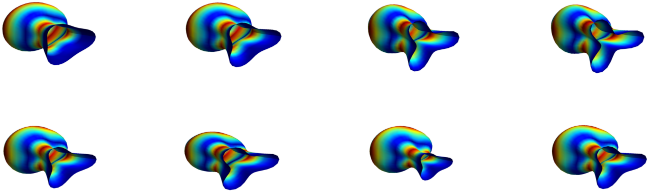


Figure: Top row: Four examples of time evolution reconstructions from the observations at 6 time points (not represented here) in the learning set. Bottom row: The simulated evolution generated from a PCA model learn from the pairs (p_0^k, u^k). The comparison between the two rows shows that the synthetised evolutions from the PCA analysis are visually good.

Higher-order using optimal transport?

Acceleration is (formally?) defined by:

$$\nabla_{\dot\rho}\dot\rho = -\nabla\cdot\left[\rho\left(v + (v,\nabla)v\right)\right], \tag{25}$$

where v is the horizontal lift associated with $\dot\rho$. Recall that

$$H(p,n) = \frac{1}{2}\langle \nabla p, \nabla p\rangle_{L^2(\rho)} = \frac{1}{2}\int_M |\nabla p|^2 \rho \, d\mu_0 \,. \tag{26}$$

From a control viewpoint, we aim at minimizing $\frac{1}{2}\int_0^1 |u|^2 \, dt$ for the controlled system:

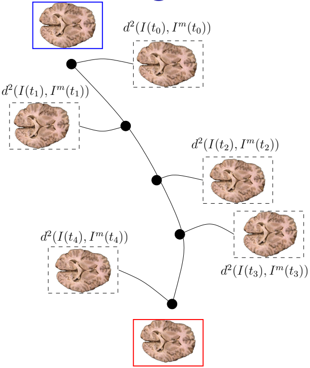
Geodesic equations:

$$\begin{cases} \dot\rho + \nabla\cdot(\rho\nabla p) = 0 \\ \dot p + \frac{1}{2}|\nabla p|^2 = u\,. \end{cases} \tag{27}$$

Splines equations:

$$\begin{cases} \dot\rho + \nabla\cdot(\rho\nabla p) = 0 \\ \dot p + \frac{1}{2}|\nabla p|^2 = u \\ P_\rho + \nabla\cdot(\rho\nabla u) = 0 \\ \dot P_p + \nabla\cdot(P_p\nabla p) - \nabla\cdot(\rho\nabla P_\rho) = 0 \\ \dot P_\rho + \nabla P_\rho\cdot\nabla p - \frac{1}{2}|\nabla u|^2 = 0\,. \end{cases} \tag{28}$$

Geodesic regression



$$\mathcal{S}(P(0)) = \frac{\lambda}{2}\langle \nabla I(0)P(0), K \star \nabla I(0)P(0)\rangle_{L^2} + \frac{1}{2}\sum_{i=1}^{k} \|I(t_i) - J_i\|_{L^2}^2 \,. \tag{29}$$

with:

$$\begin{cases} \partial_t I + v \cdot \nabla I = 0, \\ \partial_t P + \nabla \cdot (vP) = 0, \\ v + K \star (P\nabla I) = 0. \end{cases} \tag{30}$$

Adjoint equations for geodesic shooting

Proposition

The gradient of $\mathcal{S}$ is given by:
$\nabla_{P(0)}\mathcal{S} = -\hat{P}(0) + \nabla I(0) \cdot K \star (P(0)\nabla I(0))$ *where* $\hat{P}(0)$ *is given by the solution the backward PDE in time:*

$$\begin{cases} \partial_t \hat{I} + \nabla \cdot (v\hat{I}) + \nabla \cdot (P\hat{v}) = 0\,, \\ \partial_t \hat{P} + v \cdot \nabla \hat{P} - \nabla I \cdot \hat{v} = 0\,, \\ \hat{v} + K \star (\hat{I}\nabla I - P\nabla \hat{P}) = 0\,, \end{cases} \tag{31}$$

subject to the initial conditions:

$$\begin{cases} \hat{I}(1) = J - I(1)\,, \\ \hat{P}(1) = 0\,, \end{cases} \tag{32}$$

A key point: Integral formulation

Gradient descent based on an integral formulation:

Proposition

Let $I(0), J \in H^2(\Omega, \mathbb{R})$ be two images and K be a C^2 kernel on Ω. For any $P(0) \in L^2(\Omega)$, let (I, P) be the solution of the shooting equations with initial conditions $I(0), P(0)$. Then, the corresponding adjoint equations have a unique solution $(\hat{I}, \hat{P})$ in $C^0([0,1], H^1(\Omega) \times H^1(\Omega))$ such that

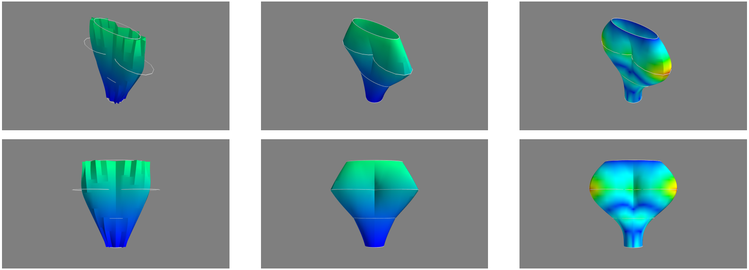
$$\begin{cases} \hat{P}(t) = \hat{P}(1) \circ \phi_{t,1} - \int_t^1 [\nabla I(s) \cdot \hat{v}(s)] \circ \phi_{t,s} \, ds \,, \\ \hat{I}(t) = \mathrm{Jac}(\phi_{t,1}) \hat{I}(1) \circ \phi_{t,1} \\ \qquad\qquad + \int_t^1 \mathrm{Jac}(\phi_{t,s}) [\nabla \cdot (P(s) \hat{v}(s))] \circ \phi_{t,s} \, ds \,. \end{cases} \tag{33}$$

with:

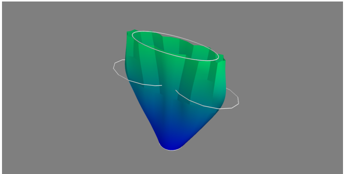
$$\begin{cases} \hat{v}(t) = K \star [P(t) \nabla \hat{P}(t) - \hat{I}(t) \nabla I(t)] \,, \\ P(t) = \mathrm{Jac}(\phi_{t,0}) P(0) \circ \phi_{t,0} \,, \\ I(t) = I(0) \circ \phi_{t,0} \,, \end{cases} \tag{34}$$

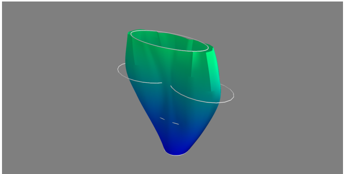
where $\phi_{s,t}$ is the flow of $v(t) = -K \star P(t) \nabla I(t)$.

Numerical examples on points



- First Column: Geodesic Regression
- Second column: Linear Interpolation
- Third Column: Spline Interpolation





Statistics on spatiotemporal data

Data is a collection of temporal sequence of shapes. Example: if **small longitudinal changes** occur: data on $T\mathcal{S}$

- Define a "static" template and transport tangent information.
- Riemannian metric on $T\mathcal{S}$ (Sasaki metric...). If $(x(t), v(t))$ is a path in $T\mathcal{S}$, the metric is given by $\|\dot{x}\|^2 + \left\|\frac{D}{Dt}v(:= w)\right\|^2$.

$$\nabla_{\dot{x}} w = 0 \text{ and } \nabla_{\dot{x}}\dot{x} + R(v, w)\dot{x} = 0\,. \tag{35}$$

Particular geodesics are given by: geodesics on M and parallel transport on TM.

Need to compare tangent spaces.

Notions of transports

Given an optimal map ϕ that maps A to B:

- Adjoint transport by diffeomorphisms: $v \to T\phi(v \circ \phi^{-1})$
- Co-adjoint transport by diffeomorphisms: $p \in T_q^* M \to g^{-1*} \cdot p \in T_{g \cdot q} M$. (momentum map equivariant)
- Parallel transport under a connection

Supervised Classification on Stable MCI and Converter MCI

Initial momentums collected for the longitudinal evolution of hippocampus: two time-points per patient.

Global / Local	**Deformation descriptor**	**Spec+ Sens**	**Spec**	**Sens**	**NPV**	**PPV**
Global	Volume difference	1.19	0.78	0.41	0.85	0.30
	Relative volume difference	1.08	0.85	0.23	0.83	0.25
Local integrated on the whole domain	Initial momentum, image transport	1.10	0.37	0.73	0.86	0.21
	Initial momentum, density transport	1.15	0.96	0.19	0.84	0.53
	Initial velocity field	1.07	0.46	0.61	0.84	0.20
Local integrated on a subregion	Initial momentum, image transport	1.18	0.63	0.55	0.86	0.25
	Initial momentum, density transport	**1.27**	**0.62**	**0.65**	**0.89**	**0.28**
	Initial velocity field	0.92	0.79	0.13	0.80	0.11
Local	Initial momentum, image transport	1.01	0.96	0.05	0.82	0.27
	Initial momentum, density transport	1.01	0.95	0.06	0.82	0.21
	Initial velocity field	0.92	0.77	0.15	0.80	0.13
Local restricted to a subregion	Initial momentum, image transport	1.10	0.68	0.42	0.84	0.23
	Initial momentum, density transport	1.17	0.68	0.49	0.85	0.26
	Initial velocity field	0.98	0.38	0.60	0.81	0.18

Local vs global descriptors of hippocampus shape evolution for Alzheimer's longitudinal population analysis, J.B Fiot et al.

Future directions

- Use of spatially varying metrics.
- Introduction of new Riemannian metrics on shapes for statistics.

H^1 optimal transport?

Geometry of Diffeomorphism Groups, Complete integrability and Geometric statistics by Khesin et al.

1. Right-invariant $\dot{H}^1$ metric on M compact descends to a metric on densities.

$$\ell(v) = \int_M \|\nabla \cdot v\|^2 \, d\mu \tag{36}$$

2. Dimension 1: H^1 metric descends to a metric on the space of densities:

$$\ell(\rho, v) = \frac{1}{2}\langle \rho v, v\rangle_{L^2} + \frac{1}{2}\langle \frac{1}{\rho}\nabla \cdot v, \nabla \cdot v\rangle_{L^2} \tag{37}$$

$\dot{H}^1$ right-invariant metric

Right-invariant metric and right action on densities.

It gives the Hellinger metric on densities:

$$d_{\dot{H}^1}(\phi, \psi) = d(\phi^* \mu, \psi^* \mu)$$
$$:= \sqrt{\mu(M)} \arccos \left(\frac{1}{\sqrt{\mu(M)}} \int_M \sqrt{\phi^* \mu \, \psi^* \mu} d\mu \right) \quad (38)$$

- Isometric to an infinite dimensional sphere.
- Flatness when $\mu(M) \to \infty$.
- Connection with the Fischer-Rao metric.