Stability of Schrödinger potentials: application to PDEs

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Motivation: Dynamical urban planning model

Solve

$$\begin{cases} \partial_t \rho_1 - \Delta \rho_1^p - \mathrm{div}(\rho_1 \nabla (V_1 + \varphi)) = 0, \\ \partial_t \rho_2 - \Delta \rho_2^q - \mathrm{div}(\rho_2 \nabla (V_2 + \psi)) = 0, \\ \rho_1(0,\cdot) = \rho_{1,0},\ \rho_2(0,\cdot) = \rho_{2,0}, \end{cases}$$

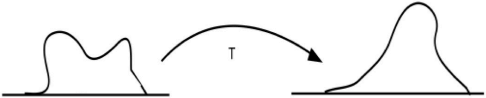
on a compact, convex subset Ω of $\mathbb{R}^n$ with no-flux boundary conditions, where $p, q \geqslant 1$, V_1 and V_2 are smooth potentials and φ and , ψ are:

- Potentials of Kantorovich: Optimal transport problem
- Potentials of Schrödinger: Entropic regularization of optimal transport problem

Plan

Table of contents

Classical Optimal Transport $\mathcal{P}(\Omega)$, $\Omega \subset \mathbb{R}^n$



- Static Monge problem (1781): given a source $\rho_1 \in \mathcal{P}(\Omega)$ and a target $\rho_2 \in \mathcal{P}(\Omega)$

$$\inf_T \left\{ \int_\Omega |x - T(x)|^2 \, d\rho_1(x) \; : \; \rho_2 = T_\# \rho_1 := \rho_1 \circ T^{-1} \right\}$$

No splitting of mass!

- Very nonlinear constraint: By a change of variable, $\rho_2 = T_\# \rho_1$ is equivalent, at least formally, to solve the Monge-Ampère equation

$$\rho_2(T(x)) \det[DT(x)] = \rho_1(x)$$

Wasserstein distance

- Kantorovich relaxation (1942):

$$W_2^2(\rho_1, \rho_2) = \min \left\{ \iint_{\Omega \times \Omega} |y - x|^2 \, d\gamma(x, y) \, : \, \gamma \in \Pi(\rho_1, \rho_2) \right\},$$

where

$$\Pi(\rho_1, \rho_2) = \left\{ \gamma \in \mathcal{P}(\Omega \times \Omega) \, : \, \pi_{1\#}\gamma = \rho_1 \text{ and } \pi_{2\#}\gamma = \rho_2 \right\}.$$

- Wasserstein distance: W_2 define a metric on $\mathcal{P}(\Omega)$ and an optimal γ is called optimal transport plan between ρ_1 and ρ_2.

Dual formulation and Brenier's theorem

Dual formulation:

$$W_2^2(\rho_1, \rho_2) = \max\left\{ \int_\Omega \varphi(x)\, d\rho_1(x) + \int_\Omega \psi(x)\, d\rho_2(x) \;:\; \varphi(x) + \psi(y) \leqslant |x-y|^2 \right\}.$$

Solutions (φ, ψ) are such that $\psi = \varphi^c := \inf_y |x-y|^2 - \varphi(y)$ and φ is called Kantorovich potential.

Theorem (Brenier 1989)

If $\rho_1 \ll \mathcal{L}_{|\Omega}$, then Kantorovich's problem admits a unique solution γ induced by a map T, i.e. $\gamma = (Id, T)_{\#}\rho_1$.
Moreover, T is the gradient of a convex function and satisfies $T = Id - \nabla\varphi$ where φ is a Kantorovich potential.

Optimal transport and labour market [Carlier-Ekeland '04]

- ρ_1, ρ_2 are the densities of inhabitants and firms in a city $\Omega \subset \mathbb{R}^2$,
- Commuting cost from x to y given by $c(x, y)$, e.g. $c(x, y) = |x - y|^2$,
- Where to work? Optimization problem over commuting cost and salary, $\psi(y)$,

$$\varphi(x) = \inf_{y \in \Omega} \{c(x, y) - \psi(y)\}.$$

- Construction of a transport map:

$$T(x) = \operatorname{argmin} \{c(x, y) - \psi(y)\}$$

  Then the equilibrium constraint reads $T_{\#}\rho_1 = \rho_2$.
- This problem is equivalent to solve the optimal transport problem

$$\inf_{\rho_2 = T_{\#}\rho_1} \int_\Omega c(T(x), x)\, d\rho_1(x),$$

  and φ and ψ are simply the Kantorovich potential associated to the dual problem.

Static urban planning model [Buttazzo-Santambrogio '05]

Optimal distribution in a city Ω: Existence and characterization of minimizers of

$$(\rho_1, \rho_2) \mapsto W_2^2(\rho_1, \rho_2) + \mathcal{F}(\rho_1) + \mathcal{G}(\rho_2),$$

where

- $\mathcal{F}$ represents a congestion effect for the inhabitants, for example

$$\mathcal{F}(\rho) = \int_\Omega F(\rho(x))\, dx = \int_\Omega \frac{F(\rho(x))}{\rho(x)} \rho(x),$$

  where F is convex and superlinear. $\frac{F(\rho)}{\rho}$ is unhappiness of a citizen living at a place with density ρ.

- $\mathcal{G}$ represents a concentration effect for the firms, for example

$$\mathcal{G}(\rho) := \iint_{\Omega\times\Omega} |x-y|^2\, d\rho(x) d\rho(y).$$

Table of contents

Wasserstein gradient flow

Energy:

$$\mathcal{E} : \mathcal{P}(\Omega) \times \mathcal{P}(\Omega) \longrightarrow (-\infty, +\infty]$$

Otto's calculus:

$$\mathrm{grad}_{W_2} \mathcal{E}(\rho_1, \rho_2) = \begin{pmatrix} -\mathrm{div}\left(\rho_1 \nabla \frac{\delta \mathcal{E}}{\delta \rho_1}(\rho_1, \rho_2)\right) \\ -\mathrm{div}\left(\rho_2 \nabla \frac{\delta \mathcal{E}}{\delta \rho_2}(\rho_1, \rho_2)\right) \end{pmatrix},$$

where $\frac{\delta \mathcal{E}}{\delta \rho}(\rho)$ is the first variation of $\mathcal{E}$.

Gradient flow of $\mathcal{E}$:

$$\begin{pmatrix} \partial_t \rho_1 \\ \partial_t \rho_2 \end{pmatrix} = -\,\mathrm{grad}_{W_2} \mathcal{E}(\rho_1, \rho_2) \qquad \Leftrightarrow \qquad \begin{cases} \partial_t \rho_1 = \mathrm{div}\left(\rho_1 \nabla \frac{\delta \mathcal{E}}{\delta \rho_1}(\rho_1, \rho_2)\right) \\ \partial_t \rho_2 = \mathrm{div}\left(\rho_2 \nabla \frac{\delta \mathcal{E}}{\delta \rho_2}(\rho_1, \rho_2)\right) \end{cases}$$

Geodesic convexity

- Let $\rho_0, \rho_1 \in \mathcal{P}(\Omega)$ and γ an optimal transport plan between ρ_0 and ρ_1. Define the Wasserstein geodesic $t \mapsto \rho_t$ by

$$\rho^t := ((1-t)\pi_1 + t\pi_2)_{\#}\gamma.$$

A functional $\mathcal{E} : (\rho_1, \ldots, \rho_l) \mapsto \mathcal{E}(\rho_1, \ldots, \rho_l)$ is said λ-geodesically convex if

$$t \in [0,1] \mapsto \mathcal{E}(\rho_1^t, \ldots, \rho_l^t) \text{ is } \lambda\text{-convex.}$$

- Examples:
  - $\mathcal{E}(\rho) = \int F(\rho)$ if F satisfies McCann's condition:

$$x \in (0, +\infty) \mapsto x^n F(x^{-n}) \text{ is convex nonincreasing}$$

  - $\mathcal{E}(\rho) = \int V\rho$ if V λ-convex

Gradient flow and geodesic convexity

Ambrosio-Gigli-Savaré, '09

Assume $\mathcal{E}$ λ-geodesically convex then

- Existence and uniqueness of the gradient flow.
- Stability: Let ρ_t, μ_t two solutions with initial conditions ρ_0, μ_0, then

$$W_2(\rho_t, \mu_t) \leq e^{-\lambda t} W_2(\rho_0, \mu_0).$$

Dynamical urban planning model

In this talk: Dynamics of

- $\mathcal{E}(\rho) = W_2^2(\rho_1, \rho_2) + \mathcal{F}(\rho_1) + \mathcal{G}(\rho_2)$

- $\mathcal{E}(\rho) = W_{2,\epsilon}^2(\rho_1, \rho_2) + \mathcal{F}(\rho_1) + \mathcal{G}(\rho_2)$

  where $W_{2,\epsilon}$ is the entropic regularization of W_2.

Table of contents

Entropic regularization

Regularized optimal transport problem:

$$\mathcal{W}_{c,\epsilon}(\rho_1, \rho_2) := \inf_{\gamma \in \Pi(\rho_1,\rho_2)} \left\{ \iint_{\Omega\times\Omega} c(x,y)\, d\gamma(x,y) + \epsilon \iint_{\Omega\times\Omega} \gamma(\log(\gamma) - 1) \right\}.$$

Can be rewritten as

$$\mathcal{W}_{c,\epsilon}(\rho_1, \rho_2) = \epsilon \inf_{\gamma \in \Pi(\rho_1,\rho_2)} \mathcal{H}(\gamma | G_\epsilon),$$

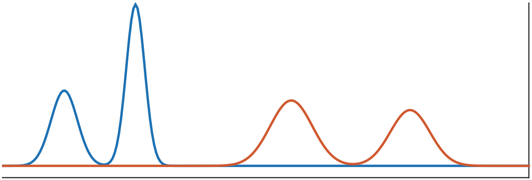
where $G_\epsilon := e^{-\frac{c}{\epsilon}}$ and $\mathcal{H}$ is the relative entropy defined by

$$\mathcal{H}(\gamma|\mu) := \begin{cases} \int_{\Omega\times\Omega} (\log\left(\frac{d\gamma}{d\mu}\right) - 1) d\gamma & \text{if } \gamma \ll \mu \\ +\infty & \text{otherwise.} \end{cases}$$

Γ-convergence: $\mathcal{W}_{c,\epsilon}$ Γ-converges to $\mathcal{W}_c$ [Léonard '12, Carlier-Duval-Peyré-Schmitzer, '17]

Addition of noise in the transport plan

[Benamou, Carlier, Cuturi, Nenna, Peyré, '15]



Marginals

Transport plans when ϵ increases

Schrödinger system

- Change of reference measure: Define

$$E(\rho_1,\rho_2) := \inf_{\gamma\in\Pi(\rho_1,\rho_2)} \left\{ \iint_{\Omega\times\Omega} c(x,y)\, d\gamma(x,y) + \epsilon\mathcal{H}(\gamma|\rho_1\otimes\rho_2) \right\}$$

Remark: $\mathcal{W}_{c,\epsilon}(\rho_1,\rho_2) = E(\rho_1,\rho_2) + \epsilon\mathcal{H}(\rho_1) + \epsilon\mathcal{H}(\rho_2)$.

- Schrödinger system: Dual solutions (ϕ_1,ϕ_2) satisfies $\rho_1\otimes\rho_2$ a.e.

$$\begin{cases} \phi_1(x) = -\epsilon\log\left(\int_\Omega e^{\frac{\phi_2(y)}{\epsilon}} G_\epsilon(x,y)\, d\rho_2(y)\right) \\ \phi_2(y) = -\epsilon\log\left(\int_\Omega e^{\frac{\phi_1(x)}{\epsilon}} G_\epsilon(x,y)\, d\rho_1(x)\right) \end{cases}$$

- same regularity as c and unique in $\tilde{\mathcal{C}}^k := \mathcal{C}^k \times \mathcal{C}^k / \sim$ where

$$(\phi_1,\phi_2) \sim (\psi_1,\psi_2) \quad \Leftrightarrow \quad \exists\kappa\in\mathbb{R} \text{ such that } \phi_1 = \psi_1 + \kappa \text{ and } \phi_2 = \psi_2 - \kappa$$

Formal gradient flow

Formally, a Wasserstein gradient flow of $\mathcal{W}_{c,\epsilon}$ satisfies

$$\begin{cases} \partial_t \rho_1 - \mathrm{div}(\rho_1 \nabla \phi_1) - \epsilon \Delta \rho_1 = 0, \\ \partial_t \rho_2 - \mathrm{div}(\rho_2 \nabla \phi_2) - \epsilon \Delta \rho_2 = 0, \\ \rho_1(0,\cdot) = \rho_{1,0},\ \rho_2(0,\cdot) = \rho_{2,0}, \end{cases}$$

where (ϕ_1, ϕ_2) are Schrödinger potentials for $E(\rho_1, \rho_2)$.

$\rightsquigarrow$ Need regularity on the Schrödinger map:

$$S : (\rho_1, \rho_2) \longmapsto (\phi_1, \phi_2)$$

Lipschitz stability of the Schrödinger map

Theorem (Carlier-Chizat-L., '22)

For $k \in \mathbb{N}^*$, assume that $c \in \mathcal{C}^{k+1}(\Omega)$. The Schrödinger map $S : \mathcal{P}(\Omega)^2 \to \tilde{\mathcal{C}}^k$ is Lipschitz continuous, i.e. there exists $C > 0$ such that, for all $(\rho_1, \rho_2), (\mu_1, \mu_2) \in \mathcal{P}(\Omega)^4$,

$$\|S(\rho_1, \rho_2) - S(\mu_1, \mu_2)\|_{\tilde{\mathcal{C}}^k} \leq C(W_2^2(\rho_1, \mu_1) + W_2^2(\rho_2, \mu_2))^{1/2}.$$

Idea of proof

Denote $\boldsymbol{\rho} = (\rho_1, \rho_2)$ and $\boldsymbol{\phi} = (\phi_1, \phi_2)$

- Rewrite the Schrödinger system as

$$F(\boldsymbol{\phi}, \boldsymbol{\rho}) = 0$$

- For any optimal transport plan $\boldsymbol{\gamma} \in \Pi(\boldsymbol{\rho}, \boldsymbol{\mu})$, consider the interpolation

$$\boldsymbol{\rho}_t = ((1-t)\pi_1 + t\pi_2)_{\#}\boldsymbol{\gamma}$$

- Apply the implicit function theorem to

$$G : \begin{array}{cl} \tilde{\mathcal{C}}^k \times [0,1] & \longrightarrow \tilde{\mathcal{C}}^k \\ (\boldsymbol{\rho}, t) & \longmapsto F(\boldsymbol{\phi}, \boldsymbol{\rho}_t) \end{array}$$

Displacement smoothness and well-posedness

Corollary (Carlier-Chizat-L., '22)

If $c \in \mathcal{C}^2$, then there exists $\lambda > 0$ such that E and $-E$ are $(-\lambda)$-displacement convex.

$\Rightarrow$ Existence and uniqueness of Wasserstein gradient flow of $\mathcal{W}_{c,\epsilon}$

$$\begin{cases} \partial_t \rho_1 - \mathrm{div}(\rho_1 \nabla \phi_1) - \epsilon \Delta \rho_1 = 0, \\ \partial_t \rho_2 - \mathrm{div}(\rho_2 \nabla \phi_2) - \epsilon \Delta \rho_2 = 0, \\ \rho_1(0,\cdot) = \rho_{1,0},\ \rho_2(0,\cdot) = \rho_{2,0}, \end{cases}$$

Asymptotic convergence

Proposition (Carlier-Chizat-L., '22)

Assume that $\mathcal{H}(\rho_i^0) < +\infty$ for every i, then $\boldsymbol{\rho}_t$, the WGF of $\mathcal{W}_{c,\epsilon}$, converges at an exponential rate to the equilibrium $\boldsymbol{\rho}^*$, defined by

$$\rho_i^*(x) = \frac{\int_\Omega e^{-c(x,y)/\epsilon} \mathrm{d}y}{\int_{\Omega^2} e^{-c/\epsilon}}$$

i.e. there exists $\kappa > 0$ independent of $\boldsymbol{\rho}^0$ such that

$$\mathcal{W}_{c,\epsilon}(\boldsymbol{\rho}_t) - \mathcal{W}_{c,\epsilon}(\boldsymbol{\rho}^*) \leq e^{-\kappa t}(\mathcal{W}_{c,\epsilon}(\boldsymbol{\rho}^0) - \mathcal{W}_{c,\epsilon}(\boldsymbol{\rho}_*)).$$

Remark: E is not λ-geodesically convex with $\lambda > 0$

Idea of proof

- Note the identities

$$E(\boldsymbol{\rho}) = \sum_{i=1}^{2} \int_\Omega S_i(\boldsymbol{\rho}) \mathrm{d}\rho_i, \quad \mathcal{W}_{c,\epsilon}(\boldsymbol{\rho}) = \epsilon \sum_{i=1}^{2} \mathcal{H}(\rho_i | e^{-S_i(\boldsymbol{\rho})/\epsilon})$$

- Using the chain rule and an integration by parts

$$\frac{d\mathcal{W}_{c,\epsilon}}{dt}(\boldsymbol{\rho}_t) = -\epsilon \left(\mathcal{I}_1(\rho_1 | e^{-S_1(\boldsymbol{\rho}_t)/\epsilon}) + \mathcal{I}_2(\rho_2 | e^{-S_2(\boldsymbol{\rho}_t)/\epsilon}) \right)$$

where $\mathcal{I}$ is the relative Fisher information

$$\mathcal{I}_i(\rho | e^{-V}) := \int_\Omega \left\| \nabla \log \left(\frac{\rho}{e^{-V}} \right) \right\|^2 \mathrm{d}\rho.$$

- Apply Log-Sobolev inequality: $\exists \kappa = \kappa(\Omega, c, \epsilon) > 0$ such that

$$\mathcal{I}_i(\rho_i | e^{-S_i(\boldsymbol{\rho}_t)/\epsilon}) \geq -\kappa \mathcal{H}(\rho_i | e^{-S_i(\boldsymbol{\rho}_t)/\epsilon}).$$

- Conclusion: Gronwall's Lemma

Extensions

- Multi-Marginal case: The existence and uniqueness can be extended to systems with more than 2 populations.

- Numerical algorithm: Based on Sinkhorn algorithm (iterate ϕ_1 and ϕ_2 in the fixed point problem)

  Previous applications: Wasserstein barycenter ([Cuturi-Doucet '14], [Benamou-Carlier-Cuturi-Nenna-Peyré '15]), Multi-marginal optimal transport ([BCCNP '15], [Nenna '16]), Wasserstein gradient flows ([Peyré '15]), etc.

Table of contents

Gradient flow of $\mathcal{E} = W_2^2(\rho_1, \rho_2) + \mathcal{F}(\rho_1) + \mathcal{G}(\rho_2)$

Since (see for example Santambrogio's book Optimal Transport for Applied Mathematicians)

$$\frac{\delta W_2^2(\cdot, \mu)}{\delta \rho}(\rho) = \varphi,$$

where φ is a Kantorovich potential i.e. belongs to

$$\left\{ \varphi \, : \, \int_\Omega \varphi \rho + \int_\Omega \varphi^c \mu = W_2^2(\rho, \mu) \right\}.$$

Formally: Wasserstein gradient flow of

$$\mathcal{E}(\rho_1, \rho_2) := W_2^2(\rho_1, \rho_2) + \sum_{i=1}^{2} \int_\Omega (\rho_i \log(\rho_i) + V_i \rho_i)$$

solves

$$\begin{cases} \partial_t \rho_1 - \Delta \rho_1 - \mathrm{div}(\rho_1 \nabla V_1) + \mathrm{div}(\rho_1 \nabla \varphi) = 0, \\ \partial_t \rho_2 - \Delta \rho_2 - \mathrm{div}(\rho_2 \nabla V_2) + \mathrm{div}(\rho_2 \nabla \psi) = 0, \end{cases} \tag{1}$$

where $(\varphi(t), \psi(t))$ is a pair of Kantorovich potential of $W_2(\rho_1(t), \rho_2(t))$, t-a.e.

Remark on geodesic convexity

- Geodesic convexity of W_2^2: if c is a $\mathcal{C}^2$ convex function such that $\partial_{i,j} c \leqslant 0$ for all $i \neq j$ in dimension 1

- Existence and uniqueness in dimension 1

- Higher dimension: W_2^2 is not geodesically convex!!
  open problem

Remark: system coupled by Monge-Ampère equation

Since $\psi = \varphi^c = \inf_y \frac{|\cdot - y|^2}{2} - \varphi(y)$, and the optimal transport map $T = Id - \nabla\varphi$ satisfies the Monge-Ampère equation,

$$\rho_2(T)\det[DT] = \rho_1.$$

Then, the system is equivalent to

$$\begin{cases} \partial_t\rho_1 - \Delta\rho_1 - \mathrm{div}(\rho_1\nabla V_1) + \mathrm{div}(\rho_1\nabla\varphi) = 0, \\ \partial_t\rho_2 - \Delta\rho_2 - \mathrm{div}(\rho_2\nabla V_2) + \mathrm{div}(\rho_2\nabla\varphi^c) = 0, \\ \rho_2(Id - \nabla\varphi)\det[I - D^2\varphi] = \rho_1, \end{cases} \tag{2}$$

where φ^c is the c-transform of φ, $\varphi^c(x) = \inf_y \frac{|x-y|^2}{2} - \varphi(y)$, and $\frac{|x|^2}{2} - \varphi$ is convex.

Weak formulation for $\mathrm{div}(\rho_1 \nabla \varphi)$

For an optimal transport plan γ and a Kantorovich potential φ we have

$$\nabla \varphi(x) = x - y \qquad \gamma - a.e.$$

Then, for all $\Phi \in \mathcal{C}_c^\infty([0, +\infty] \times \mathbb{R}^n)$,

$$\int_0^{+\infty} \int_\Omega \nabla \Phi(t,x) \cdot \nabla \varphi(t,x) \rho_1(t,x)\, dxdt$$
$$= \int_0^{+\infty} \int_{\Omega \times \Omega} \nabla \Phi(t,x) \cdot \nabla \varphi(t,x)\, d\gamma(t,x,y)\, dt$$
$$= \int_0^{+\infty} \int_{\Omega \times \Omega} \nabla \Phi(t,x) \cdot (x-y)\, d\gamma(t,x,y)\, dt.$$

where $\gamma(t)$ is the optimal transport plan for $W_2(\rho_1(t), \rho_2(t))$, t-a.e.

The nonlinear term $\rho_1 \nabla \varphi$ is replaced by a linear term in γ

Weak solutions

A weak solution of (2) is a curve $t \in (0,+\infty) \mapsto (\rho_1(t),\rho_2(t)) \in \mathcal{P}^{ac}(\Omega)^2$ such that $\rho_i \in L^1((0,T);W^{1,1}(\Omega))$ for all $T < +\infty$ and for all $\Phi \in \mathcal{C}_c^\infty([0,+\infty] \times \mathbb{R}^n)$,

$$\int_0^{+\infty} \bigg(\int_\Omega \partial_t \Phi \rho_1 \, dx - \int_\Omega \nabla \Phi \cdot (\nabla \rho_1 + \nabla V_1 \rho_1) \\ - \int_{\Omega\times\Omega} (x-y)\cdot \nabla\Phi(t,x)\, d\gamma(t,x,y) \bigg) \, dt = - \int_\Omega \Phi(0,x)\rho_{1,0}(x)\, dx,$$

and

$$\int_0^{+\infty} \bigg(\int_\Omega \partial_t \Phi \rho_2 \, dx - \int_\Omega \nabla \Phi \cdot (\nabla \rho_2 + \nabla V_2 \rho_2) \\ - \int_{\Omega\times\Omega} (y-x)\cdot \nabla\Phi(t,y)\, d\gamma(t,x,y) \bigg) \, dt = - \int_\Omega \Phi(0,x)\rho_{2,0}(x)\, dx,$$

where $\gamma(t)$ is the optimal transport plan for $W_2(\rho_1(t),\rho_2(t))$, t-a.e.

Existence Theorem

Theorem (L. '20)

Assume that $\rho_{1,0}$ and $\rho_{2,0}$ have finite Entropy, then system (2) admits at least one weak solution.

JKO scheme ([De Giorgi, '93],[Jordan, Kinderlehrer, Otto,'98], [Ambrosio, Gigli, Savaré, '05] : Given $h > 0$, construct (ρ_1^k, ρ_2^k) by induction

$$(\rho_1^{k+1}, \rho_2^{k+1}) = \operatorname{argmin} \sum_{i=1}^{2} \frac{1}{2h} W_2^2(\rho_i, \rho_i^k) + \mathcal{E}_1(\rho_1) + \mathcal{E}_2(\rho_2) + W_2^2(\rho_1, \rho_2). \quad (3)$$

where

$$\mathcal{E}_i(\rho) = \begin{cases} \int_\Omega \rho \log(\rho) + V_i \rho & \text{if } \rho \ll \mathcal{L}, \\ +\infty & \text{otherwise,} \end{cases}$$

$\rho_{i,h}$: piecewise interpolation in time of $(\rho_i^k)_k$.

Extensions (1)

More than 2 populations and more general cost functions: Multi-Marginal Optimal Transport problems

$$W_2^2(\rho_1, \rho_2) \rightsquigarrow \mathcal{W}_c(\rho_1, \dots, \rho_I) = \inf_{\gamma \in \Pi(\rho_1, \dots, \rho_I)} \int_{\Omega^I} c(x_1, \dots, x_I)\, d\gamma.$$

Application to gradient flow of Wasserstein Barycenter [Agueh-Carlier 2010]

Extensions (2)

Different transport problems for each population: Semi-implicit JKO scheme (introduced by [DiFrancesco and Fagioli '14])

$$\rho_i^{k+1} \in \mathrm{argmin}_\rho \frac{1}{2h} W_2^2(\rho, \rho_i^k) + 2h(\mathcal{E}_i(\rho) + \mathcal{W}_{c_i}^k(\rho)),$$

where,

$$\mathcal{W}_{c_i}^k(\rho) = \mathcal{W}_{c_i}(\rho_1^k, \dots, \rho_{i-1}^k, \rho, \rho_{i+1}^k, \dots, \rho_I^k).$$

Main argument: $\mathcal{W}_{c_i}^k$ is Lipschitz in the Wasserstein space $\Rightarrow$ time compactness.

Table of contents

Open problems and future works

- Uniqueness or not in dimension $d \geqslant 2$ for the non regularized problem with smooth initial data
- Γ-Convergence of the gradient flow of the entropic regularization problem to the gradient flow of the optimal transport problem: Serfaty [Γ-convergence of gradient flows on Hilbert and metric spaces and applications] or send ϵ and h to 0 in the same time during the JKO procedure
- Extension to different transport problem: taking into account the traffic congestion during the transport [Carlier-Santambrogio, '05] or a ring road going arround the city [Monsaingeon, '21]
- Asymptotic behaviour for the unregularized problem

Thank you for your attention!