Non-decreasing martingale coupling

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Conference: New Monge Problems and Applications

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Outline

The Optimal Transport problem

Let $\mu, \nu \in \mathcal{P}(\mathbb{R}^d)$ and $c : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ measurable. The **Optimal Transport (OT)** problem states as:

$$\mathrm{OT}(\mu, \nu, c) := \inf_{\pi \in \Pi(\mu,\nu)} \int_{\mathbb{R}^d \times \mathbb{R}^d} c(x, y)\, \pi(dx, dy)$$

where $\Pi(\mu, \nu)$ denotes the set of couplings between μ and ν, that is

$$\Pi(\mu, \nu) = \{\pi \in \mathcal{P}(\mathbb{R}^d \times \mathbb{R}^d) \mid \pi(dx, \mathbb{R}^d) = \mu(dx) \text{ and } \pi(\mathbb{R}^d, dy) = \nu(dy)\}.$$

Let $\Pi_M(\mu,\nu)$ be the set of **martingale coupling** between μ and ν,

$$\Pi_M(\mu,\nu) = \left\{\pi(dx,dy) = \mu(dx)\pi_x(dy) \in \Pi(\mu,\nu) \mid \mu(dx)\text{-a.e., } \int_{y\in\mathbb{R}} y\,\pi_x(dy) = x\right\}.$$

The **Martingale Optimal Transport (MOT)** problem states as:

$$\mathrm{MOT}(\mu,\nu,c) = \inf_{\pi\in\Pi_M(\mu,\nu)} \int_{\mathbb{R}^d\times\mathbb{R}^d} c(x,y)\,\pi(dx,dy).$$

The Martingale Optimal Transport problem

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The **Martingale Optimal Transport (MOT)** problem states as:

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For traders who want to deal an exotic option that pays $c(S_{T_1}, S_{T_2})$ where $S_{T_1} \sim \mu$ and $S_{T_2} \sim \nu$, the robust bounds are provided by

$$\mathrm{MOT}(\mu,\nu,c) \le \mathbb{E}[c(S_{T_1}, S_{T_2})] \le -\mathrm{MOT}(\mu,\nu,-c).$$

According to Hobson and Neuberger [4], the robust price bounds for the forward start options is $-\mathrm{MOT}(\mu,\nu,-|S_{T_1} - S_{T_2}|)$.

Strassen Theorem, 1965 [6]

Let $\mu, \nu \in \mathcal{P}(\mathbb{R}^d)$ be with finite first moment, then

$$\begin{aligned}\Pi^M(\mu,\nu) \neq \emptyset &\iff \forall \varphi : \mathbb{R}^d \to \mathbb{R} \text{ convex}, \int_{\mathbb{R}^d} \varphi(x)\,\mu(dx) \leq \int_{\mathbb{R}^d} \varphi(y)\,\nu(dy) \\ &\iff \mu \leq_{cx} \nu\end{aligned}$$

The MOT problem was introduced by

- Beiglböck, Henry-Labordère, and Penkner [1] in discrete time setting in 2013,
- Galichon, Henry-Labordère, and Touzi [3] in continuous time setting in 2014
- many contributions since : Acciaio, Alfonsi, Backhoff-Veraguas, Bayraktar, Beiglböck, Brückerhoff, Corbetta, Cox, De March, Galichon, Ghoussoub, Guo, Guyon, Henry-Labordère, Hobson, Huesmann, Juillet, Kim, Lim, Neufeld, Nutz, Oblój, Pagès, Pammer, Sester, Siorpaes, Stebegg, Tan, Touzi,...

Maximization of the cost function $c(x, y) = |x - y|$

Hobson and Neuberger [4, Theorem 8.2] state that there exists a maximizing martingale coupling

$$\pi^{\mathrm{HN}} = \int_0^1 \left(\frac{r(u) - q(u)}{r(u) - p(u)} \delta_{(q(u), p(u))} + \frac{q(u) - p(u)}{r(u) - p(u)} \delta_{(q(u), r(u))} \right) du$$

where p, q, r are **non-decreasing** on $(0, 1)$ with $p \leq q \leq r$.

When μ is continuous **[1]**,

$$\pi^{\mathrm{HN}} = \int_{\mathbb{R}} \left(\frac{g(x) - x}{g(x) - f(x)} \delta_{(x, f(x))} + \frac{x - f(x)}{g(x) - f(x)} \delta_{(x, g(x))} \right) \mu(dx)$$

for **non-decreasing** functions f and g such that $\forall x \in \mathbb{R}$, $f(x) \leq x \leq g(x)$.

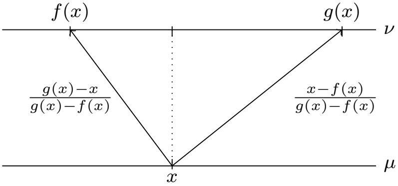

Figure: The non-decreasing coupling π^{HN}

[1] Beiglböck and Juillet stated the uniqueness of π^{HN} in [2, Theorems 7.3]

Question - Is π^{HN} still optimal for other costs?

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Answer - Yes from our numerical experiments! For $\mu \leq_{cx} \nu$,

$$\begin{aligned} \pi^{\text{HN}} \quad \text{attains} \quad & \overline{\mathcal{M}}^{\rho}_{\rho}(\mu,\nu) = \sup_{\pi \in \Pi_{\text{M}}(\mu,\nu)} \int_{\mathbb{R}\times\mathbb{R}} |x-y|^{\rho}\, \pi(dx,dy) \text{ when } \rho \in (0,2), \\ \text{attains} \quad & \underline{\mathcal{M}}^{\rho}_{\rho}(\mu,\nu) = \inf_{\pi \in \Pi_{\text{M}}(\mu,\nu)} \int_{\mathbb{R}\times\mathbb{R}} |x-y|^{\rho}\, \pi(dx,dy) \text{ when } \rho > 2. \end{aligned}$$

Normal distribution with $\mu \sim \mathcal{N}(0, 0.24)$ and $\nu \sim \mathcal{N}(0, 0.28)$					
ρ	$\overline{\mathcal{M}}_\rho$	$\int \lvert x-y\rvert^\rho \pi^{\mathrm{HN}}(dx,dy)$	ρ	$\underline{\mathcal{M}}_\rho$	$\int \lvert x-y\rvert^\rho \pi^{\mathrm{HN}}(dx,dy)$
0.3	0.578737	0.578734	2.1	0.023250	0.023250
0.7	0.280626	0.280625	2.3	0.016369	0.016370
1.0	0.163683	0.163683	2.5	0.011537	0.011538
1.4	0.080147	0.080146	3.0	0.004834	0.004836
1.9	0.033062	0.033062	5.0	0.000158	0.000159

Log-normal distribution with $\mu \sim$ Lognormal$(0, 0.24)$ and $\nu \sim$ Lognormal$(0, 0.28)$					
ρ	$\overline{\mathcal{M}}_\rho$	$\int \lvert x-y\rvert^\rho \pi^{\mathrm{HN}}(dx,dy)$	ρ	$\underline{\mathcal{M}}_\rho$	$\int \lvert x-y\rvert^\rho \pi^{\mathrm{HN}}(dx,dy)$
0.3	0.560822	0.560818	2.1	0.019848	0.019848
0.7	0.261948	0.261947	2.3	0.013871	0.013871
1.0	0.149069	0.149069	2.5	0.009718	0.009718
1.4	0.070960	0.070960	3.0	0.004036	0.004037
1.9	0.028471	0.028471	5.0	0.000139	0.000140

Binomial distribution with $\mu \sim B(10, 0.5)$ and $\nu \sim B(40, 0.5)$					
ρ	$\overline{\mathcal{M}}_\rho$	$\int \lvert x-y\rvert^\rho \pi^{\mathrm{HN}}(dx,dy)$	ρ	$\underline{\mathcal{M}}_\rho$	$\int \lvert x-y\rvert^\rho \pi^{\mathrm{HN}}(dx,dy)$
0.3	1.295262	1.295258	2.1	6.684936	6.685331
0.7	1.840491	1.840486	2.3	8.099506	8.101478
1.0	2.407133	2.407133	2.5	9.831921	9.837066
1.4	3.466126	3.466100	3.0	16.092828	16.120046
1.9	5.527885	5.527669	5.0	129.174431	130.930978

Poisson distribution with $\lambda_\mu = 1$ and $\lambda_\nu = 4$					
ρ	$\overline{\mathcal{M}}_\rho$	$\int \lvert x-y\rvert^\rho \pi^{\mathrm{HN}}(dx,dy)$	ρ	$\underline{\mathcal{M}}_\rho$	$\int \lvert x-y\rvert^\rho \pi^{\mathrm{HN}}(dx,dy)$
0.3	1.163662	1.163638	2.1	3.735438	3.735438
0.7	1.452421	1.452421	2.3	4.363190	4.363190
1.0	1.740083	1.740083	2.5	5.121420	5.121420
1.4	2.256078	2.256078	3.0	7.806860	7.807091
1.9	3.213878	3.213878	5.0	56.008318	56.008318

Question - Why $\rho = 2$ is a threshold?

Answer - When $\mu, \nu \in \mathcal{P}_2(\mathbb{R})$, for all $\pi \in \Pi_M(\mu, \nu)$, by the martingale property $\int_{\mathbb{R}} y \, \pi_x(dy) = x$, we have

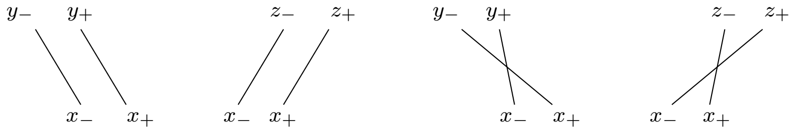
$$\begin{aligned}\int_{\mathbb{R}^2} |x-y|^2 \pi(dx, dy) &= \int_{\mathbb{R}} y^2 \nu(dy) + \int_{\mathbb{R}} x^2 \mu(dx) - 2 \int_{\mathbb{R}} x \underbrace{\int_{\mathbb{R}} y \, \pi_x(dy)}_{x} \mu(dx) \\ &= \int_{\mathbb{R}} y^2 \nu(dy) - \int_{\mathbb{R}} x^2 \mu(dx) \\ &= \underline{\mathcal{M}}_2^2(\mu, \nu) = \overline{\mathcal{M}}_2^2(\mu, \nu).\end{aligned}$$

Definition 1

Let $\mu, \nu \in \mathcal{P}_1(\mathbb{R})$ be such that $\mu \leq_{cx} \nu$. A martingale coupling $\pi^{\uparrow} \in \Pi_M(\mu, \nu)$ is called **non-decreasing** if there exists a Borel set $\Gamma \subseteq \mathbb{R} \times \mathbb{R}$ such that $\pi(\Gamma) = 1$ and

(a) if $(x_-, y_-), (x_+, y_+) \in \Gamma$ with $y_- \leq x_-$, $y_+ \leq x_+$, and $x_- < x_+$, then $y_- \leq y_+$

(b) if $(x_-, z_-), (x_+, z_+) \in \Gamma$ with $x_- \leq z_-$, $x_+ \leq z_+$, and $x_- < x_+$, then $z_- \leq z_+$.



(a) non-decreasing (left) (b) non-decreasing (right) (c) forbidden case (left) (d) forbidden case (right)

Figure: Non-decreasing martingale coupling

Note: π^{HN} is a non-decreasing martingale coupling as well.

Proposition 1

Let $\mu, \nu \in \mathcal{P}_1(\mathbb{R})$ be such that $\mu \leq_{cx} \nu$ and $\varphi : \mathbb{R}_+ \to \mathbb{R}$ be some increasing concave function.

(i) *If $\tilde{\pi} \in \Pi_M(\mu, \nu)$ maximizes $\int \varphi(|x-y|)\pi(dx, dy)$ over $\pi \in \Pi_M(\mu, \nu)$, then $\tilde{\pi}$ is* ***non-decreasing****. When φ is continuous, there exists such a maximizing coupling.*

(ii) *If μ and φ are continuous, there exists a unique* ***non-decreasing*** *martingale coupling $\tilde{\pi}$ that maximizes $\int \varphi(|x-y|)\pi(dx, dy)$.*

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(ii) *If μ and φ are continuous, there exists a unique* ***non-decreasing*** *martingale coupling $\tilde{\pi}$ that maximizes $\int \varphi(|x-y|)\pi(dx, dy)$.*

For $\rho \in (0, 1]$,

- since $\mathbb{R}_+ \ni z \mapsto z^\rho$ is concave and increasing, a martingale coupling that maximizes $\int |x-y|^\rho \, \pi(dx, dy)$ is **non-decreasing**;
- we can construct an example such that any martingale coupling is **non-decreasing** and the maximizing coupling actually depends on ρ.

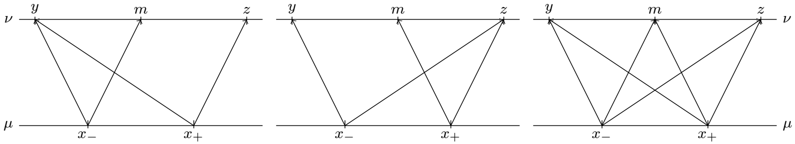

Figure: Different non-decreasing martingale couplings for $\rho \in (0, 1]$

Definition 2 (Nested supports condition)

We say that $\mu, \nu \in \mathcal{P}_1(\mathbb{R})$ satisfy the nested supports condition if there exist $-\infty < a \leq b < +\infty$ such that $\mu([a, b]) = 1$ and $\nu((a, b)) = 0$.

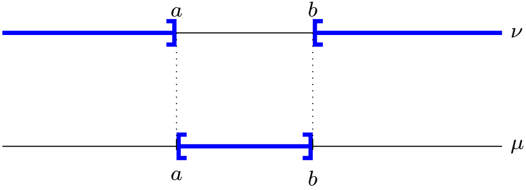

Figure: The nested supports condition

Proposition 2

Let $\mu, \nu \in \mathcal{P}_1(\mathbb{R})$ be such that $\mu \leq_{cx} \nu$. Under the nested supports condition, there <u>*exists a unique*</u> ***non-decreasing*** *martingale coupling $\pi^{\uparrow} \in \Pi_M(\mu, \nu)$.*

Since the set $\{\eta \in \mathcal{P}_1(\mathbb{R}) : \int_{\mathbb{R}} z\eta(dz) = \int_{\mathbb{R}} y^2\nu(dy) - \int_{\mathbb{R}} x^2\mu(dx)\}$ is a complete lattice with minimal element $\delta_{\int_{\mathbb{R}} y^2\nu(dy) - \int_{\mathbb{R}} x^2\mu(dx)}$ by Kertz and Röler [5],

Proposition 3

Let $\mu, \nu \in \mathcal{P}_2(\mathbb{R})$ be such that $\mu \leq_{cx} \nu$. Then, the set $\left\{|x-y|^2 \# \pi : \pi \in \Pi_M(\mu,\nu)\right\}$ admits an infimum for the convex order.

[2] Beiglböck and Juillet [2, Theorems 7.3]

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When there exists $\underline{\pi} \in \Pi_M(\mu,\nu)$ such that $|x-y|^2\#\underline{\pi}$ is equal to the **infimum** for the convex order,

$$\int_{\mathbb{R}^2} |x-y|^\rho \underline{\pi}(dx,dy) = \begin{cases} \overline{\mathcal{M}}_\rho^\rho(\mu,\nu) & \text{when} \quad \rho \in (0,2), \\ \underline{\mathcal{M}}_\rho^\rho(\mu,\nu) & \text{when} \quad \rho > 2. \end{cases}$$

In particular, when $\nu(dy) = \int_{x\in\mathbb{R}} \frac{1}{2}\left(\delta_{x-a}(dy) + \delta_{x+a}(dy)\right)\mu(dx)$ for some $a \in \mathbb{R}$, the **infimum** is equal to δ_{a^2} and

$$\underline{\pi}(dx,dy) = \frac{1}{2}\left(\delta_{x-a}(dy) + \delta_{x+a}(dy)\right)\mu(dx).$$

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When there exists $\underline{\pi} \in \Pi_M(\mu,\nu)$ such that $|x-y|^2\#\underline{\pi}$ is equal to the **infimum** for the convex order,

$$\int_{\mathbb{R}^2} |x-y|^\rho \underline{\pi}(dx,dy) = \begin{cases} \overline{\mathcal{M}}_\rho^\rho(\mu,\nu) & \text{when} \quad \rho \in (0,2), \\ \underline{\mathcal{M}}_\rho^\rho(\mu,\nu) & \text{when} \quad \rho > 2. \end{cases}$$

In particular, when $\nu(dy) = \int_{x\in\mathbb{R}} \frac{1}{2}\left(\delta_{x-a}(dy) + \delta_{x+a}(dy)\right)\mu(dx)$ for some $a \in \mathbb{R}$, the **infimum** is equal to δ_{a^2} and

$$\underline{\pi}(dx,dy) = \frac{1}{2}\left(\delta_{x-a}(dy) + \delta_{x+a}(dy)\right)\mu(dx).$$

When μ is continuous, the uniqueness **[2]** of the maximizing couplings for $\rho = 1$ implies that the non-decreasing coupling π^{HN} is the only possible value for $\underline{\pi}$.

[2] Beiglböck and Juillet [2, Theorems 7.3]

Example of inf for the convex order

For $p \in (0,1)$, $y_- < y_+ < x_- < x_+ < z_- < z_+$, we set $\mu = p\delta_{x_-} + (1-p)\delta_{x_+}$,

$$\nu = p\frac{z_- - x_-}{z_- - y_-}\delta_{y_-} + p\frac{x_- - y_-}{z_- - y_-}\delta_{z_-} + (1-p)\frac{z_+ - x_+}{z_+ - y_+}\delta_{y_+} + (1-p)\frac{x_+ - y_+}{z_+ - y_+}\delta_{z_+}.$$

Let

$$x_+ - y_- \geq z_- - x_-, \qquad z_+ - x_- \geq x_+ - y_+ \text{ and}$$

$$(x_- - y_-) \wedge (x_+ - y_+) \wedge (z_- - x_-) \wedge (z_+ - x_+) \geq (x_- - y_+) \vee (z_- - x_+),$$

note that this setting is under the nested supports condition. There exists $\pi^\uparrow$ with

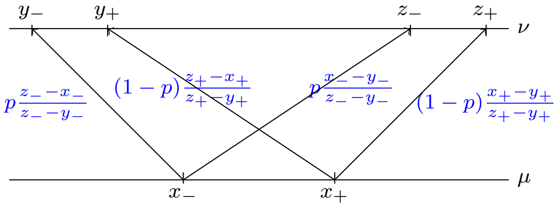


Figure: A natural non-decreasing martingale coupling $\pi^\uparrow \in \Pi_M(\mu,\nu)$

and we have $|x-y|^2\#\pi^\uparrow(dx,dy) = \inf_{cx}\{|x-y|^2\#\pi : \pi \in \Pi_M(\mu,\nu)\}$.

Proposition 3

Let $\mu \leq_{cx} \nu$ *be such that there exist* $\underline{y} < \overline{y} < \underline{x} < \overline{x} < \underline{z} < \overline{z}$ *with* $\mu([\underline{x}, \overline{x}]) = 1$ *and* $\nu\left([\underline{y}, \overline{y}] \cup [\underline{z}, \overline{z}]\right) = 1$*. If*

$$\underline{x} - \overline{y} \geq \alpha_\rho(\overline{z} - \overline{y}),$$

$$\underline{z} - \overline{x} \geq \alpha_\rho(\underline{z} - \underline{y}),$$

where $\alpha_\rho \in (0, \frac{1}{2})$ *is the unique solution of* $\psi_\rho(\alpha) = 0$ *with* $\psi_\rho : (0, 1] \to \mathbb{R}$ *defined by*

$$\psi_\rho(\alpha) = \alpha + \alpha^{2-\rho}(1-\alpha)^{\rho-1} + 1 - \rho.$$

Then, the <u>*unique*</u> *non-decreasing coupling* $\pi^\uparrow$ *in* $\Pi_M(\mu, \nu)$ *is the* <u>*unique*</u> *optimal coupling in* $\Pi_M(\mu, \nu)$ *that attains* $\overline{\mathcal{M}}_\rho(\mu, \nu)$ *when* $\rho \in (1, 2)$ *and* $\underline{\mathcal{M}}_\rho(\mu, \nu)$ *when* $\rho > 2$*.*

α_ρ

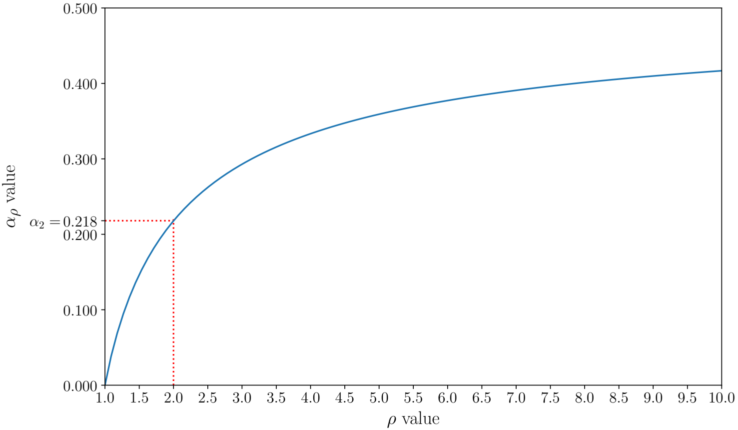


Figure: The function $\rho \mapsto \alpha_\rho$ (with α_ρ computed by a root finding algorithm).

Non-preservation of the optimality

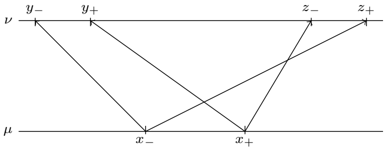
For $p \in (0,1)$ and $y_- < y_+ < x_- < x_+ < z_- < z_+$, we set $\mu = p\delta_{x_-} + (1-p)\delta_{x_+}$

$$\nu = p\frac{z_+ - x_-}{z_+ - y_-}\delta_{y_-} + p\frac{x_- - y_-}{z_+ - y_-}\delta_{z_+} + (1-p)\frac{z_- - x_+}{z_- - y_+}\delta_{y_+} + (1-p)\frac{x_+ - y_+}{z_- - y_+}\delta_{z_-}.$$

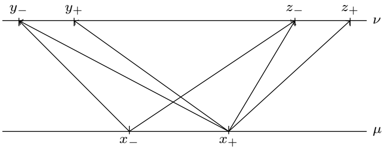
The couplings

$$\pi^\star = p\left(\frac{z_+ - x_-}{z_+ - y_-}\delta_{(x_-,y_-)} + \frac{x_- - y_-}{z_+ - y_-}\delta_{(x_-,z_+)}\right) + (1-p)\left(\frac{z_- - x_+}{z_- - y_+}\delta_{(x_+,y_+)} + \frac{x_+ - y_+}{z_- - y_+}\delta_{(x_+,z_-)}\right).$$

belong to $\Pi_M(\mu,\nu)$.



(a) $\pi^\star$



(b) non-decreasing coupling $\pi^\uparrow$

Figure: Comparison between $\pi^\star$ and $\pi^\uparrow$

Proposition 4

Let $\rho \in (1,2) \cup (2,+\infty)$ and $y_- < y_+ < z_- < z_+$ be such that $z_- - y_- > z_+ - z_-$ and $z_- - y_+ \geq (\rho - 1)^{\frac{1}{2-\rho}}(z_+ - z_-)$. For $\rho \in (1,2)$ (resp. $\rho > 2$), there exists $x_\rho \in (y_+, z_-)$ such that for $x_\rho < x_- < x_+ < z_-$, $\pi^\star$ is the unique optimal coupling that attains $\overline{\mathcal{M}}_\rho(\mu,\nu)$ (resp. $\underline{\mathcal{M}}_\rho(\mu,\nu)$).

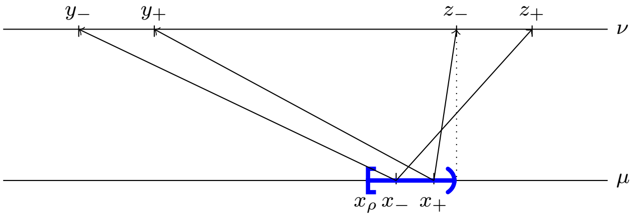


Figure: The x_ρ condition on $\pi^\star$

Decomposition of martingale couplings

For $\pi \in \Pi_M(\mu, \nu)$, we denote by

$$\nu_l^\pi(dy) = \int_{x \in \mathbb{R}} \mathbb{1}_{\{y<x\}} \pi(dx, dy),$$
$$\nu_0^\pi(dy) = \int_{x \in \mathbb{R}} \mathbb{1}_{\{y=x\}} \pi(dx, dy),$$
$$\nu_r^\pi(dy) = \int_{x \in \mathbb{R}} \mathbb{1}_{\{y>x\}} \pi(dx, dy).$$

Note that $\nu_l^\pi + \nu_0^\pi + \nu_r^\pi = \nu$. For (ν_l, ν_r) a couple of non-negative measures such that $\nu_l + \nu_r \leq \nu$, we denote

$$\Pi_M(\mu, \nu, \nu_l, \nu_r) = \{\pi \in \Pi_M(\mu, \nu) : (\nu_l^\pi, \nu_0^\pi, \nu_r^\pi) = (\nu_l, \nu - \nu_l - \nu_r, \nu_r)\}.$$

Theorem 1

Let $\mu, \nu \in \mathcal{P}_1(\mathbb{R})$ be such that $\mu \leq_{cx} \nu$ and $\mu \neq \nu$. Let (ν_l, ν_r) a couple of non-negative measures such that $\nu_l + \nu_r = \nu$. We have

$$\Pi_M(\mu, \nu, \nu_l, \nu_r) \neq \emptyset \Longleftrightarrow \exists! \pi^\uparrow \in \Pi_M(\mu, \nu, \nu_l, \nu_r) \text{ non-increasing},$$

and then, for each continuous increasing concave function $\varphi : \mathbb{R}_+ \to \mathbb{R}$,

$$\forall \pi \in \Pi_M(\mu, \nu, \nu_l, \nu_r) \backslash \{\pi^\uparrow\}, \int_{\mathbb{R}^2} \varphi(|y-x|)\pi(dx, dy) < \int_{\mathbb{R}^2} \varphi(|y-x|)\pi^\uparrow(dx, dy). \quad (1)$$

Corollary

For (ν_l, ν_r) a couple of non-negative measures such that $\nu_l + \nu_r \leq \nu$, we have

$$\Pi_M(\mu, \nu, \nu_l, \nu_r) \neq \emptyset \Longleftrightarrow \exists!\, \pi^\uparrow \in \Pi_M(\mu, \nu, \nu_l, \nu_r)$$

such that $\dfrac{\pi^\uparrow(dx, dy) + (\nu_l + \nu_r - \nu)(dx)\delta_x(dy)}{\nu_l(\mathbb{R}) + \nu_r(\mathbb{R})}$ is non-decreasing and for all $\pi \in \Pi_M(\mu, \nu, \nu_l, \nu_r) \setminus \{\pi^\uparrow\}$, equation (1) above still holds.

Sketch of proof

Let $\phi_\pi : [0,1] \to [0, \nu_l(\mathbb{R})]$ be such that

$$\phi_\pi(F_\mu(x)) = \int_{\mathbb{R}^2} \mathbb{1}_{\{z<y\leq x\}} \pi(dy, dz) \in [0, \nu_l(\mathbb{R})].$$

and we also have

$$F_\mu(x) - \phi_\pi(F_\mu(x)) = \int_{\mathbb{R}^2} \mathbb{1}_{\{y \leq x, y<z\}} \pi(dy, dz) \in [\nu_r(\mathbb{R}), 1].$$

Let $\tilde{\nu}_l = \frac{\nu_l}{\nu_l(\mathbb{R})}$, $\tilde{\nu}_r = \frac{\nu_r}{\nu_r(\mathbb{R})}$ and for $u \in [0,1]$ and $v \in [(u - \nu_r(\mathbb{R}))^+, u \wedge \nu_l(\mathbb{R})]$,

$$G(u,v) = \int_0^v F_{\tilde{\nu}_l}^{-1}\left(\frac{w}{\nu_l(\mathbb{R})}\right) dw + \int_0^{u-v} F_{\tilde{\nu}_r}^{-1}\left(\frac{w}{\nu_r(\mathbb{R})}\right) dw.$$

Let $\phi_\uparrow : [0,1] \to [0, \nu_l(\mathbb{R})]$ be a non-decreasing function such that $u \mapsto u - \phi_\uparrow(u)$ is also non-decreasing and we need to show

$$\forall u \in [0,1],\ \phi_\uparrow(u) \in [(u - \nu_r(\mathbb{R}))^+, u \wedge \nu_l(\mathbb{R})] \text{ and } \int_0^u F_\mu^{-1}(w) dw = G(u, \phi_\uparrow(u)). \quad (2)$$

Steps:

1. The existence of $\phi_\uparrow(F_\mu(x)) \in [(F_\mu(x) - \nu_r(\mathbb{R}))^+, \phi_\pi(F_\mu(x))]$ for $x \in \mathbb{R}$.
2. The monotonicity of $x \mapsto \phi_\uparrow(F_\mu(x))$ and $x \mapsto F_\mu(x) - \phi_\uparrow(F_\mu(x))$.
3. The monotonicity of $u \mapsto \phi_\uparrow(u)$ and $u \mapsto u - \phi_\uparrow(u)$ on $(0,1)$.
4. The existence of $\pi^\uparrow \in \Pi_M(\mu, \nu, \nu_l, \nu_r)$ such that $\phi_{\pi^\uparrow} = \phi_\uparrow$; the uniqueness of $\pi^\uparrow$.
5. The optimality of $\pi^\uparrow$ for $\varphi : \mathbb{R}_+ \to \mathbb{R}$ continuous increasing and concave.

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