Monge problem on the real line approached from concave cost problems.

New Monge Problems and Applications

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Outline

1. Introduction on the MK-transport problems on $\mathbb{R}$

2. The quadratic transport problem $p > 1$ and $p = 1^+$

3. The “concave case” $p < 1$ and finally $p = 1^-$

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Transport problem

Le $\mu, \nu \in \mathcal{P}(\mathbb{R})$. We denote by $\mathsf{Marg}(\mu, \nu)$ the space of transport plans.

Transport problem

Minimize

$$f_p : \pi \in \mathsf{Marg}(\mu, \nu) \mapsto \iint |y - x|^p \mathrm{d}\pi(x, y).$$

The original Monge problem

In 1978 Monge formulated the problem in the following setting :

- The “déblai" and “remblai" are sets ; the transports are maps
- The déblai and remblai are in $\mathbb{R}^2$ of $\mathbb{R}^3$.
- The cost is $c(x,y) = |y-x|^p$ for $p = 1$.

Our “new” Monge problem

Our problem is not so far :

- The “déblai" and “remblai" are measures ; the transports are plans
- The déblai and remblai are in $\mathbb{R}^1$.
- The cost is $c(x, y) = |y - x|^p$ for $p = 1 - \varepsilon$ or in fact $p = 1^-$.

Different values of p : $p = 1$

Transport problem

Minimize

$$f_1 : \pi \in \mathsf{Marg}(\mu, \nu) \mapsto \iint |y - x| \mathrm{d}\pi(x, y).$$

Characterized by the non-uniqueness of the solution
even for "nice" mesures (Ex : shelf of books).

Different values of p : $p = 0$

Transport problem

Minimize

$$f_0 : \pi \in \mathsf{Marg}(\mu, \nu) \mapsto \iint |y - x|^0 \mathrm{d}\pi(x, y).$$

(Here $0^0 = 0$ and $x^0 = 1$ for $x > 0$).
The minimal value of f_0 is the *total variation* of μ and ν.

Different values of p : $p = -1$

Transport problem

Minimize

$$f_{-1} : \pi \in \mathsf{Marg}(\mu, \nu) \mapsto \iint \frac{1}{|y - x|} \mathrm{d}\pi(x, y).$$

(Here $0^{-1} = +\infty$).
Studied in
Cotar, Friesecke, Klüppelberg (2013) *Density functional theory and optimal transportation with Coulomb cost.*
Buttazzo, De Pascale, P. Gori-Giorgi (2012) *Optimal-transport formulation of electronic density-functional theory.*

Different values of p : $p = 2$ and $p > 1$

Transport problem

Minimize

$$f_2 : \pi \in \mathsf{Marg}(\mu, \nu) \mapsto \iint |y - x|^2 \mathrm{d}\pi(x, y).$$

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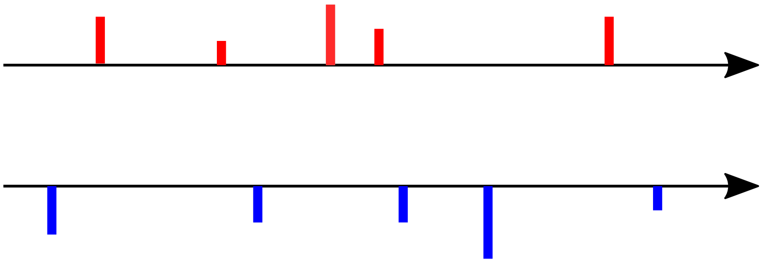
Reference theorem

Theorem

Let $p > 1$ and μ, ν be such that $\min\{f_p(\pi) | \ \pi \in \mathrm{Marg}(\mu, \nu)\} < +\infty$.
For every $\pi \in \mathrm{Marg}(\mu, \nu)$ the following statements are equivalent :

1. π is a minimizer of f_p.
2. there exists $R \subset \mathbb{R}^2$ such that $\pi(R) = 1$ and
$$(x, y), (x', y') \in R \quad \text{implies} \quad x < x' \Rightarrow y \leq y'.$$
3. $\pi = (F_\mu^{-1}, F_\nu^{-1})_\# \mathcal{L}^1$.

Transporting without crossing



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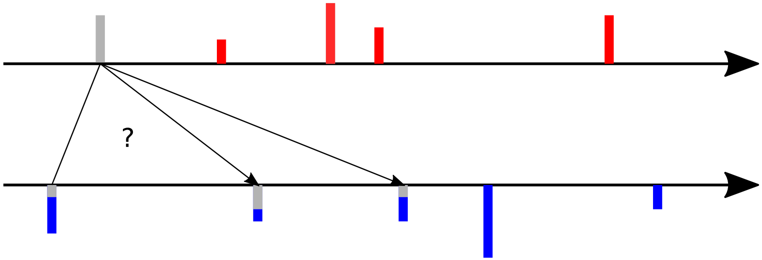

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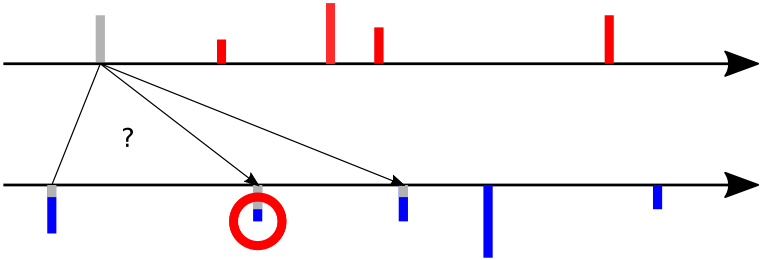

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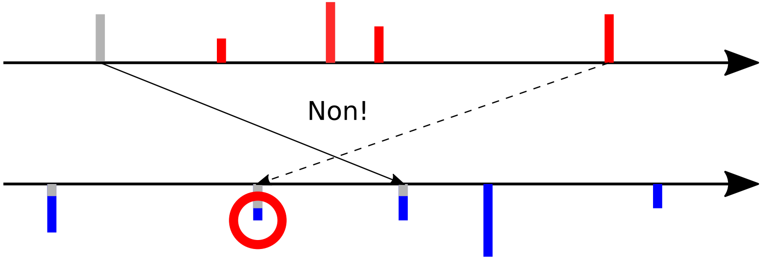

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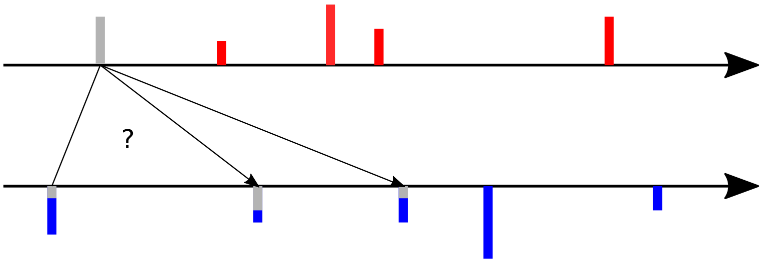

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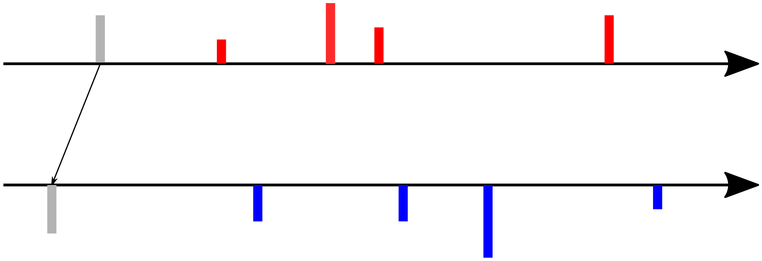

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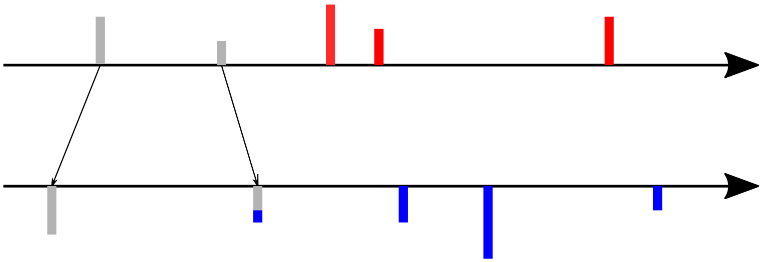

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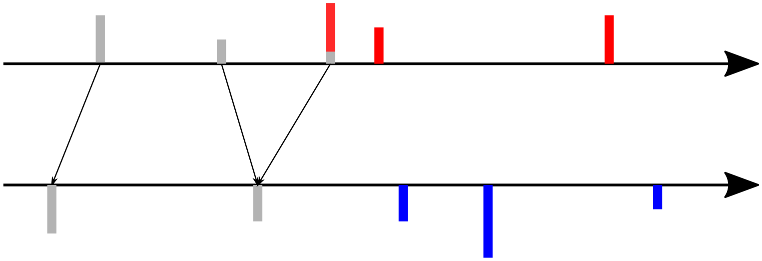

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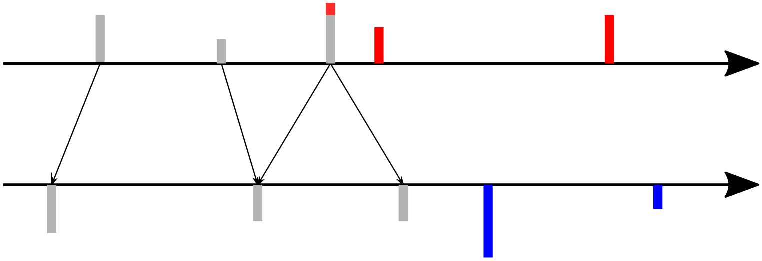

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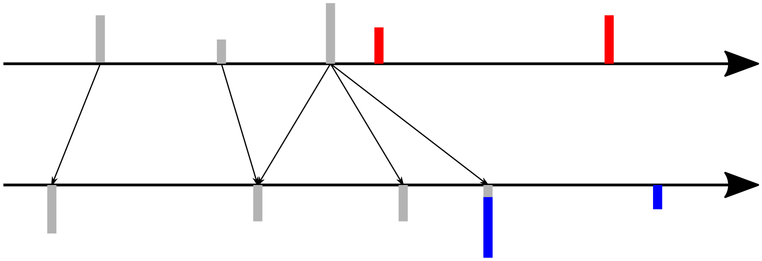

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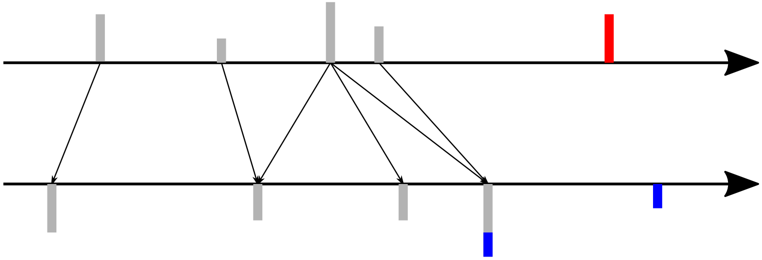

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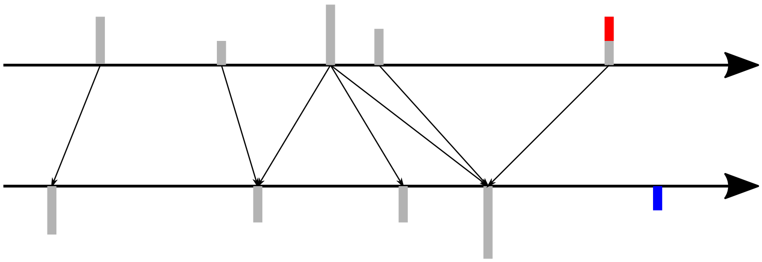

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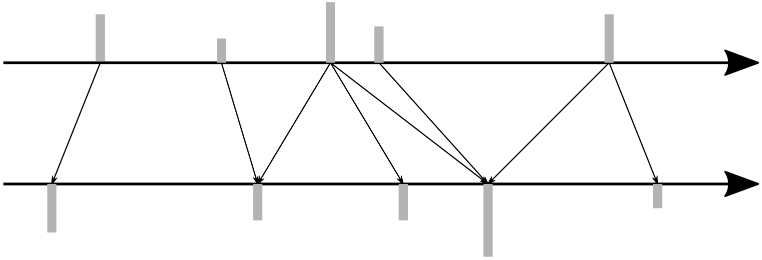

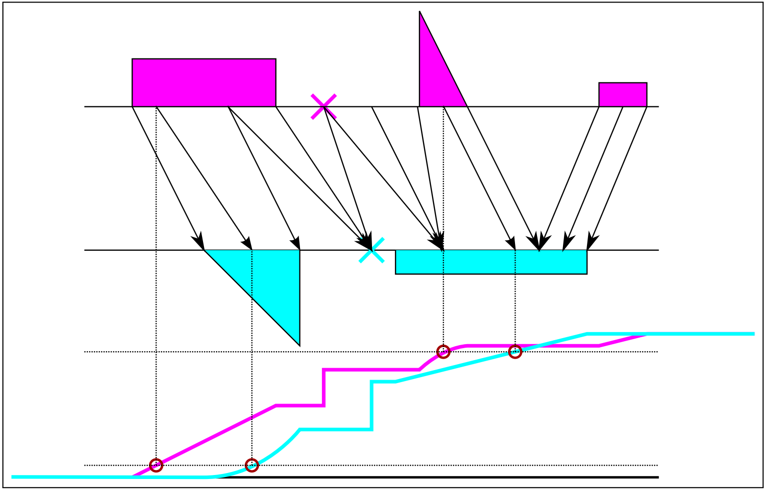

Remarks

- No regularity assumption on μ and ν.
- The optimal transport plan does not depend on $p \in (1, +\infty)$.
- It is uniquely determined.

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Some literature around $p \in (0, 1)$

For points :

- *Statistics of noncoding RNAs : alignment and secondary structure prediction.* Nechaev, Tamm, Valba (2011)
- *Local matching indicators for transport problems with concave costs.* Delon, Salomon, Sobelovski (2012)
- *The Dyck bound in the concave 1-dimensional random assignment model.* Caracciolo, D'Achille, Erba, Sportiello (2020)

For continuous measures :

- *The geometry of optimal transportation.* Gangbo, McCann (1996)
- *Exact solutions to the transportation problem on the line.* McCann (1999)

Citation of Gangbo–McCann (1996)

[...] the concavity of the cost function favours a long trip and a shorter trip over two trips of average length ; as a result, it can be efficient for two trucks carrying the same commodity to pass each other travelling opposite directions on the highway [...]. In optimal solutions, such 'pathologies" may nest on many scales, leading to a natural hierarchy among the regions [...].

Remarks

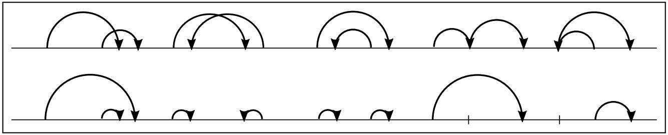
- Uniqueness can happen if μ or ν is continuous.
- The optimal transport plan is not unique.
- The set of optimizers depends on $p \in (0, 1)$.

Theorem (J. 2020)

Theorem

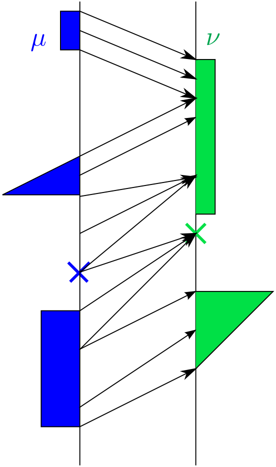
Let μ, ν have finit 1st moment and $\pi \in \mathrm{Marg}(\mu, \nu)$. Let $q < 1$.
The following are equivalent :

1. **Solution** L^{1-} There exists a sequence $(\pi_n, p_n)_n$ such that $\pi_n \to \pi$ where π_n is a minimizer of f_{p_n} and $p_n \nearrow 1$.
2. **Solution** $L^{1,q}$ π minimizes f_1 and, minimizes f_q among the minimizers.
3. **Monotonicity** The route pairs on the top are forbidden :



4. **Simulation** π is the "excursion coupling".

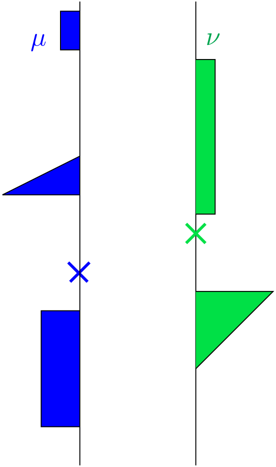
Example with pictures



Quantile coupling

optimal for $c_p(x,y) = |y-x|^p$, $p > 1$.

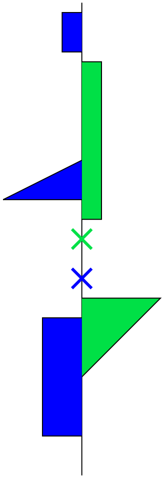
Example with pictures



Excursion coupling

"optimal" for $c_p(x,y) = |y-x|^p$, $p \to 1^-$.

Example with pictures



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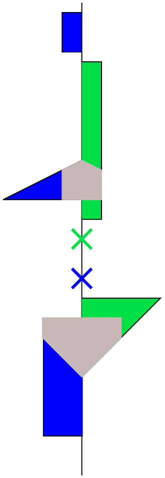

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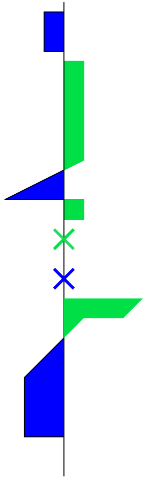

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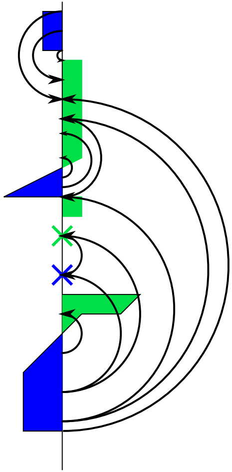

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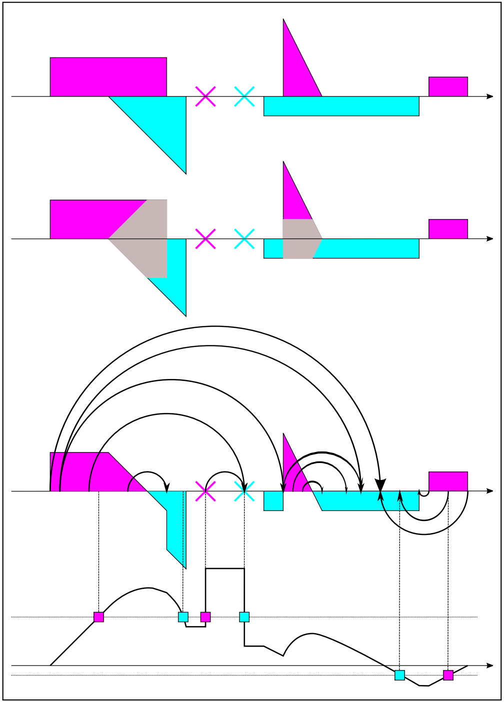

Final remarks and questions

Remark :

- Dyck paths and parentheses
- Other approach of the selection by Di Marino and Louet (2018)
  The entropic regularization of the Monge problem on the real line.

Questions :

- Algorithms in $\mathbb{R}^d$ $(d > 1)$ for the concave case $(p < 1)$?
- What happens in $\mathbb{R}^d$ for $d > 1$ and $p = 1^-$?
- Check the continuous tree approach.
- Other limit theorems for $p = p_0^+$ and $p = p_0^-$?